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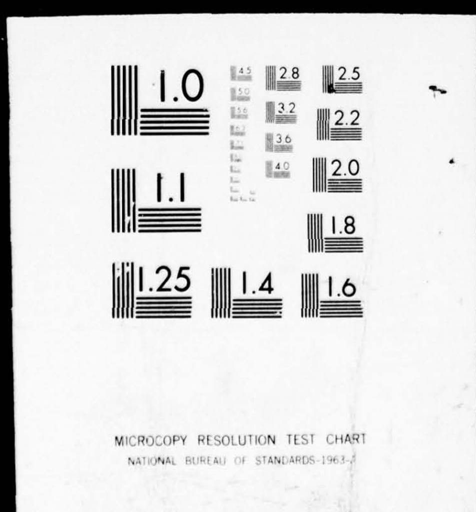

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CALCULATION OF TEMPERATURE FIELDS IN CONCRETE HYDRAULIC STRUCTURES

Sh. N. Plyat

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Methods are presented for calculation of the temperature fields in concrete water engineering structures during the period of construction and of use. The book is intended for scientific workers and engineers--hydraulic engineers and hydroelectric power engineers--and may also be useful for teachers and students in these specialty areas.

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CALCULATION OF TEMPERATURE FIELDS IN CONCRETE HYDRAULIC STRUCTURES

Sh. N. Plyat

Raschety Temperaturnykh Poley Betonnykh Gidrosooruzheniy,
Moscow, Energiya Press, 1974

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FOREWORD

In planning and constructing any concrete water engineering structure, a great deal of attention must be given to regulation of its temperature mode; in most cases, problems of regulation of the temperature mode become predominant in the selection of the type of dam and technology of its construction. Therefore, the development of methods for calculation of the temperature fields in water engineering structures is a significant problem.

Calculation of the temperature fields is the determining step in the establishment of the temperature of joining of construction seams, selection and demonstration of efficient methods of cooling of concrete poured, necessary to achieve the desired temperatures by the planned moment of closure of the structure and its individual parts, for the creation of normal temperature conditions for functioning of the drainage system, for the development of measures assuring favorable conditions for setting of the concrete in the winter in various parts of the dam, etc.

Determination of temperature fields as a component part of prediction of temperature and shrinkage stresses in concrete structures is an important area of investigation, helping to assure crack resistance.

At the present time, in connection with the rapid development and introduction to practice of new methods of computational mathematics at scientific, planning and construction organizations, there are many engineers who have a rather high level of mathematical training. As a result of this, most calculations are performed by computer. We are speaking here not only and not so much of the fact that the calculation methods which were formerly suggested for "manual" calculations are now being translated into machine language (although this is a positive phenomenon). The primary trend in this process of restructuring of computation is the application of new calculation methods, which both consider and effectively utilize the peculiarities and advantages of electronic computers. Under these conditions, the statement and solution of problems can be made to correspond to actual conditions of construction and utilization of structures.

The primary goal of this book is to provide a systematic presentation of the methods of calculation of temperature fields in concrete water engineering structures.

The solutions presented in the book are given in the form of algorithms convenient for programming for computers (in computer addresses or in ALGOL). Many of them have been produced in the form of computer programs and are currently in practical use.

The content of the book is clear from the table of contents.

The material is presented in a manner intended to teach the reader with mathematical training at the level presented in technical higher educational institutions to state and solve problems related to the determination of the temperature mode of concrete structures. However, it is hardly necessary to read the book straight through. The main chapters of the book are written to be independent so that the calculation methods presented in each chapter can be applied in practice using the materials of that chapter alone.

The book is intended for engineers (scientific workers, planners and builders) working on problems of regulation of the temperature mode of concrete water engineering and other similar structures, for graduate students and senior students in water engineering, construction, power engineering, thermal engineering and other specialties at universities to allow a deeper study of the corresponding areas of their educational programs.

The author considers it his pleasant duty gratefully to acknowledge the significant aid and support so generously provided him at the All-Union Scientific Research Institute for Hydraulic Engineering imeni V. Ye. Vedeneyev.

All comments and questions should be addressed to: 113114, Moscow, M-114, Shlyuzovaya Nab., 10, Energiya Press.

The Author

CHAPTER 1. PRINCIPLES OF THE THEORY OF HEAT CONDUCTIVITY OF SOLIDS

1-1. General Dependences

Temperature Field

The task of the theory of heat conductivity is to determine the temperature field of a body. Here, by temperature field, we understand the set of values of temperatures in the space being studied at a given moment in time.

If the temperature field changes with time, i.e., if the temperature function which describes it is as follows:

$$T = T(x_1, x_2, x_3, \tau),$$

where x_j ($j = 1, 2, 3$) are the coordinates; τ is time, it is called unstable; otherwise, i.e., if

$$T = T(x_1, x_2, x_3),$$

it is called stable.

Graphically, a temperature field is imaged by means of isothermal surfaces -- surfaces, on all points of which the temperature is identical. A set of isotherms -- lines of identical temperature -- are produced by planes intersecting the isothermal surfaces. Within a body, the isothermal surfaces (and isotherms) do not intersect and are not interrupted, either closing within the body or ending at the boundary of the body.

The concept of the temperature gradient is introduced to characterize the intensity of change in temperature.

The temperature gradient refers to the vector directed perpendicularly to the isothermal surface in the direction of increasing temperature, and is numerically equal to the partial derivative of temperature with respect to this direction.

The general expression for the temperature gradient is

$$\operatorname{grad} T = \frac{\partial T}{\partial n} \mathbf{n}^0, \tag{1-1}$$

where n^0 is the unit vector of the normal to the isothermal surface.

The temperature gradient is equal to:

in rectangular coordinates x, y, z

$$\operatorname{grad} T = \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k};$$

in cylindrical coordinates r, ϕ, z

$$\operatorname{grad} T = \frac{\partial T}{\partial r} \mathbf{e}_r + \frac{\partial T}{\partial z} \mathbf{e}_z + \frac{1}{r} \frac{\partial T}{\partial \varphi} \mathbf{e}_\varphi;$$

in spherical coordinates r, θ, ϕ

$$\operatorname{grad} T = \frac{\partial T}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial T}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial T}{\partial \varphi} \mathbf{e}_\varphi.$$

Here i, j, k, e_r, e_ϕ, e_θ and e_z are unit base vectors.

Fourier's Law of Heat Conductivity

The basic law of heat conductivity is Fourier's law. For isotropic bodies it is formulated as follows: the heat flux density vector at each point in the temperature field is proportional to the temperature gradient at that point

$$\mathbf{W} = -\lambda \operatorname{grad} T. \tag{1-2}$$

The heat flux density vector W is directed perpendicularly to the isothermal surface in the direction of decreasing temperature and is numerically equal to the quantity of heat passing through a unit area of the isothermal surface in a unit time.

The proportionality factor λ is called the heat conductivity coefficient.

The heat flux density w at a given point in the field through any surface is equal to the projection of the heat flux density vector W on the direction of the perpendicular ν to this surface in the direction of decreasing temperature, i.e.,

$$w = W_\nu = -\lambda \frac{\partial T}{\partial \nu}.$$

Differential Equation of Heat Conductivity

The temperature field of a body is defined by solving the Fourier heat conductivity equation. This equation in differential form relates the temperature function to the space-time coordinates. For a heterogeneous isotropic body it is:

$$c\gamma \frac{\partial T}{\partial \tau} = \mathrm{div}\,(\lambda\, \mathrm{grad}\, T) + q\,(x_1, x_2, x_3, \tau, T). \tag{1-3}$$

Here q is the power of internal heat sources, i.e., the quantity of heat liberated (or absorbed) by internal sources (or sinks) per unit volume of the body per unit time; λ is the heat conductivity coefficient; c is the specific heat capacity of the substance; γ is the density of the substance.

The differential operation div b (divergence of vector b) has the following form:

in rectangular coordinates

$$\mathrm{div}\,\mathbf{b} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z};$$

in cylindrical coordinates

$$\mathrm{div}\,\mathbf{b} = \frac{1}{r} \cdot \frac{\partial}{\partial r}(r b_r) + \frac{\partial b_z}{\partial z} + \frac{1}{r} \frac{\partial b_\varphi}{\partial \varphi};$$

in spherical coordinates

$$\mathrm{div}\,\mathbf{b} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 b_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\, b_\theta) + \frac{1}{r \sin\theta} \frac{\partial b_\varphi}{\partial \varphi},$$

b_x, b_y, b_z, b_r, b_ϕ, b_θ are the components of vector b with respect to the corresponding axes.

If the heat conductivity coefficient λ is independent of coordinates and temperature, then the differential equation for heat conductivity is written as follows:

$$\frac{\partial T}{\partial} = a\nabla^2 T + \frac{1}{c\gamma} q. \tag{1-4}$$

Here a is the temperature conductivity coefficient, $a = \lambda/c\gamma$; $\nabla^2 T =$ $= \text{div grad } T$ is the Laplacian of temperature, the differential operation defined by the expressions:

in rectangular coordinates

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2};$$

in cylindrical coordinates

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2};$$

in spherical coordinates

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2}.$$

For a stable temperature field in an isotropic homogeneous body with internal heat sources

$$\nabla^2 T = -\frac{q}{\lambda} \quad \text{(Poisson's equation)}; \qquad (1\text{-}5)$$

or without heat sources

$$\nabla^2 T = 0 \quad \text{(Laplace equation)}. \qquad (1\text{-}6)$$

The differential equations presented above for unstable heat conductivity (1-3) and (1-4) are parabolic type equations. If λ, c, γ and q are arbitrary functions of coordinates and time, while q, possibly, is also a linear function of temperature, then equation (1-3) is linear (more precisely quasilinear) and its integration is quite difficult. If λ, c, γ and q are arbitrary functions of the coordinates and time, while q, possibly, is also a linear function of temperature, then equation (1-3) is linear.

In what follows, unless we specifically state otherwise, we will be analyzing linear differential equations for unstable heat conductivity in rectangular and cylindrical coordinates, of the following form

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + p(\tau)T + Q(x_1, x_2, x_3, \tau). \qquad (1\text{-}7)$$

where a is a constant.

The functions $p(\tau)$ and $Q(x_1, x_2, x_3, \tau)$ in particular cases may be either constant or equal to 0.

If $Q \neq 0$, equation (1-7) is called heterogeneous, otherwise, where $Q = 0$ -- homogeneous.

Initial and Boundary Conditions

Fourier's differential equation for unstable heat conductivity is an equation in partial derivatives. It has an infinite set of solutions. Unambiguous description of the process of heat conductivity requires that a unique solution to this equation be distinguished. This is achieved by assigning the initial and boundary conditions (the so-called edge conditions).

The initial condition characterizes the thermal state of the body at a certain fixed moment in time $\tau = \tau_0$, usually taken as the 0 moment $\tau_0 = 0$, and is fixed either as a function of the coordinates

$$T(x_1, x_2, x_3, \tau)\,|_{\tau=0} = T(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3), \tag{1-8}$$

or as a constant

$$T(x_1, x_2, x_3, 0) = T_0 = \text{const}. \tag{1-9}$$

The basic types of boundary conditions are:

1. The boundary condition of the first kind -- the temperature $\phi(\mathcal{T}, \tau)$ is fixed at the contour Γ of the body

$$T(x_1, x_2, x_3, \tau)\,|_{\Gamma} - \varphi(\mathcal{P}, \tau). \tag{1-10}$$

2. The boundary condition of the second kind -- the heat flux $\eta(\mathcal{T}, \tau)$ is fixed at the contour Γ of the body

$$\left.\frac{\partial T(x_1, x_2, x_3, \tau)}{\partial n}\right|_{\Gamma} = \frac{1}{\lambda}\eta(\mathcal{P}, \tau); \tag{1-11}$$

A particular case (generally used as the condition of symmetry)

$$\left.\frac{\partial T(x_1, x_2, x_3, \tau)}{\partial n}\right|_\Gamma = 0. \qquad (1\text{-}12)$$

3. The boundary condition of the third kind -- the heat exchange of the body with the environment is Newtonian, and the ambient temperature $\psi(\mathcal{T}, \tau)$ is fixed

$$\left.\frac{\partial T(x_1, x_2, x_3, \tau)}{\partial n}\right|_\Gamma = h[\psi(\mathcal{P}, \tau) - T(x_1, x_2, x_3, \tau) \mid_\Gamma]. \qquad (1\text{-}13)$$

The Newton law of heat exchange[1], based on which the boundary condition of the third kind is defined, establishes the proportional dependence of heat flux density on the surface of the body as it cools as a function of the temperature difference between the surface and the environment, i.e.,

$$W_n = \alpha(\psi - T_n). \qquad (1\text{-}14)$$

In formulas (1-10)-(1-13): $\mathcal{T}$ is a point on the surface Γ of the body; n is an external perpendicular to the surface Γ of the body at point $\mathcal{T}$; $h = \alpha/\lambda$ is the relative heat transfer coefficient; α is the heat transfer coefficient; ϕ, η, ψ are assigned temperatures of the surface (boundary conditions of the first kind), the heat flux at the surface (boundary conditions of the second kind), the ambient temperature (boundary conditions of the third kind) respectively; in formula (1-14), furthermore, T_n is the temperature of the surface during the process of heat exchange.

As an approximation, it is usually assumed that the relative heat transfer coefficient h is independent of temperature and time and is identical over defined sectors of the surface of the body.

The boundary conditions of the first kind -- formulas (1-10) and of the second kind -- formula (1-12) can be looked upon as limiting versions of the boundary condition of the third kind, namely: as $h \to \infty$ in the first case and as $h \to 0$ in the second.

4. The conjugation conditions -- conditions of equality of temperature and heat fluxes on the line of conjugation of two bodies with different heat-

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[1] This law is also called the law of convective heat exchange or the law of convection.

physical characteristics[1] [with ideal thermal contact[2]]

$$T_1(\mathcal{S}, \tau) = T_2(\mathcal{S}, \tau); \; \lambda_1 \frac{\partial T_1(\mathcal{S}, \tau)}{\partial n} = \lambda_2 \frac{\partial T_2(\mathcal{S}, \tau)}{\partial n}, \tag{1-15}$$

n is the common normal to the contact surface at point $\mathcal{T}$.

As we can easily see, boundary conditions of the first, second and third kinds can be represented by a single expression

$$\alpha \frac{\partial T}{\partial n}\Big|_{\Gamma} + \beta T\,|_{\Gamma} = \gamma g(\mathcal{S}, \tau), \tag{1-16}$$

where α, β, γ are constants.

Where $g \neq 0$, boundary conditions such as (1-16) are called heterogeneous, where $g = 0$ they are called homogeneous. Similarly, the initial condition (1-8) is heterogeneous if $f \neq 0$, homogeneous if $f = 0$.

1-2. Statement of the Problem of Heat Conductivity

Formulation of the Problem of Heat Conductivity

In this book, we shall analyze isotropic, homogeneous and piecewise-homogeneous bodies[3].

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[1] The heat-physical characteristics of a substance include the temperature conductivity coefficient a, heat conductivity coefficient λ and specific heat capacity $c\gamma$.

[2] In the case of nonideal thermal contact, the conjugation conditions are written as

$$\lambda_1 \frac{\partial T_1(\mathcal{S}, \tau)}{\partial n} = \frac{1}{R}\left[T_2(\mathcal{S}, \tau) - T_1(\mathcal{S}, \tau)\right],$$
$$\lambda_1 \frac{\partial T_1(\mathcal{S}, \tau)}{\partial n_1} = \lambda_2 \frac{\partial T_2(\mathcal{S}, \tau)}{\partial n}, \tag{1-15'}$$

where R is the thermal contact resistance. Conjugation conditions of this type (1-15') will not be analyzed in this book.

[3] A piecewise-homogeneous body is a body consisting of a finite number of areas with different heat-physical characteristics, though the characteristics are constant within the limits of each individual area.

For isotropic homogeneous bodies, the problem of heat conductivity is formulated as follows: find the temperature function $T(x_1, x_2, x_3, \tau)$, defined and continuous within the closed area $-R_1 \le x_1 \le R_2$, $-L_1 \le x_2 \le L_2$, $-D_1 \le x_3 \le D_2$, $0 \le \tau \le t$, satisfying in the open area $-R_1 < x_1 < R_2$, $-L_1 < x_2 < L_2$, $-D_1 < x_3 < D_2$, $0 < \tau \le t$ the differential equation

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + pT + Q$$
$$(-R_1 < x_1 < R_2,\ -L_1 < x_2 < L_2,\ -D_1 < x_3 < D_2,\ 0 < \tau \leq t), \tag{1-17}$$

the initial condition

$$T(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3)$$
$$(-R_1 \leqslant x_1 \leqslant R_2,\ -L_1 \leqslant x_2 \leqslant L_2,\ -D_1 \leqslant x_3 \leqslant D_2), \tag{1-18}$$

and the boundary conditions

$$\alpha \left.\frac{\partial T(x_1, x_2, x_3, \tau)}{\partial n}\right|_{\Gamma} + \beta T(x_1, x_2, x_3, \tau)\,|_{\Gamma} = \gamma g(\mathscr{P}, \tau). \tag{1-19}$$

The formulation of the problem of heat conductivity of piecewise continuous bodies differs from that presented above in that the differential equation for heat conductivity (1-17) and initial condition (1-18) are replaced with systems of differential equations and initial conditions (equal to the number of areas with different heat-physical characteristics), while the boundary conditions are supplemented by the conjugation conditions [such as (1-15)] at the contact surfaces.

Note 1. In both formulations of the problem of heat conductivity, it is assumed that the temperature function $T(x_1, x_2, x_3, \tau)$ must satisfy the differential equation in the open area, but not where $\tau = 0$ and not on the contour of the body. If, for example, it is required that T satisfy the heat conductivity equation where $\tau = 0$ as well, this would lead to the necessity of imposing extremely rigid conditions on function $f(x_1, x_2, x_3)$, namely: $f(x_1, x_2, x_3)$ would have to have second derivatives with respect to the coordinates; this would greatly constrict the range of physical phenomena which could be analyzed.

Note 2. The condition of continuity $T(x_1, x_2, x_3, \tau)$ in the closed area should be understood as a limiting condition: upon approach to the space-time boundaries of the area studied, the temperature function $T(x_1, x_2, x_3, \tau)$ approaches the assigned initial and boundary conditions, i.e.,

$$\lim_{\tau \to 0} T(x_1, x_2, x_3, \tau) \to f(x_1, x_2, x_3)$$

for a fixed point (x_1, x_2, x_3) in the area $(-R_1 < x_1 < R_2, -L_1 < x_2 < L_2, -D_1 < x_3 < D_2)$;

$$\lim_{x_1, x_2, x_3 \to \Gamma} \alpha \frac{\partial T(x_1, x_2, x_3, \tau)}{\partial n} + \lim_{x_1, x_2, x_3 \to \Gamma} \beta T(x_1, x_2, x_3, \tau) \to \gamma g(\mathcal{P}, \tau)$$

for a fixed $\tau > 0$ ($\mathcal{P}$ is a point on contour Γ of the body).

In this book, we utilize the simplest classification of temperature fields and, correspondingly, of heat conductivity problems, based on their division into one-dimensional, two-dimensional and three-dimensional fields.

A temperature field is called one-dimensional if the temperature function depends only on one spatial coordinate and time.

In rectangular coordinates with a one-dimensional temperature field

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0,$$

the Laplacian

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} \tag{1-20}$$

and the temperature function

$$T = T(x, \tau).$$

One-dimensional problems are also encountered in analysis of planar temperature fields in a cylindrical system of coordinates with axial symmetry.

For such problems

$$\frac{\partial T}{\partial \varphi}=\frac{\partial T}{\partial z}=0,$$

the Laplacian

$$\nabla^2 T=\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r} \qquad (1\text{-}21)$$

and the temperature function

$$T=T(r,\ \tau).$$

As we can easily see, for a one-dimensional field, the Laplacian $\nabla^2 T$ can be represented by the single formula

$$\nabla^2 T=\frac{1}{\xi^i}\,\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right), \qquad (1\text{-}22)$$

where $\xi = x$, $i = 0$ in rectangular coordinates and $\xi = r$, $i = 1$ in cylindrical coordinates.

For two-dimensional temperature fields, it is characteristic that the temperature function depends on two spatial coordinates and time.

In a rectangular system of coordinates these are planar temperature fields, for which

$$\frac{\partial T}{\partial z}=0,$$

the Laplacian

$$\nabla^2 T=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} \qquad (1\text{-}23)$$

and the temperature function

$$T=T(x,\ y,\ \tau).$$

In a cylindrical system of coordinates with axial symmetry these are solid temperature fields for which

$$\frac{\partial T}{\partial \varphi}=0,$$

the Laplacian

$$\nabla^2 T=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2} \quad (1\text{-}24)$$

and the temperature function

$$T=T(r,\ z,\ \tau).$$

For two-dimensional temperature fields, the Laplacian $\nabla^2 T$ in rectangular and cylindrical (with axial symmetry) coordinates can be represented by the formula

$$\nabla^2 T=\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right)+\frac{\partial^2 T}{\ ^2}, \quad (1\text{-}25)$$

where $\xi = x$, $\zeta = y$, $i = 0$ in rectangular coordinates; $\xi = r$, $\zeta = z$, $i = 1$ in cylindrical coordinates.

Finally, three-dimensional temperature fields are described by temperature functions which depend on three spatial coordinates and time.

In this book, we study three-dimensional temperature fields only in rectangular systems of coordinates. Consequently, in this case we will deal with the temperature function

$$T=T(x,\ y,\ z,\ \tau)$$

and the Laplacian

$$\nabla^2 T=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}. \quad (1\text{-}26)$$

Basic Properties of Solutions

1. If in the heat conductivity equation the free term

$$Q(x_1,\ x_2,\ x_3,\ \tau)\geqslant 0,$$

then the least value of the solution, i.e., the temperature function $T(x_1, x_2, x_3, \tau)$ is reached either where $\tau = 0$ or at the boundary of the body.

2. If in the heat conductivity equation the free term

$$Q(x_1, x_2, x_3, \tau) \leqslant 0,$$

then the maximum value of temperature function $T(x_1, x_2, x_3, \tau)$ is achieved either where $\tau = 0$ or at the boundary of the body.

3. If the heat conductivity equation is homogeneous ($Q \equiv 0$), the greatest and least value of $T(x_1, x_2, x_3, \tau)$ is reached either where $\tau = 0$ or at the boundary of the body.

4. The solution of the heat conductivity problem depends continually both on the edge conditions (initial and boundary) and on the free term in the differential equation.

The first three properties are sometimes called the maximum (or minimum) principle, the fourth property defines the correctness of the statement of the heat conductivity problem -- slight changes in assigned functions included in the edge conditions and in the right portion of the equation corresponding to slight changes in the solution.

Uniqueness of the Solution of the Heat Conductivity Problem

Up to the present time, no proof has been produced of the existence and uniqueness of the solution of the heat conductivity problem in general form. Proof of the existence of a solution is a purely mathematical problem. For our purposes, it is sufficient to accept the possibility of physical realization of the process as the condition of existence of a solution.

In theory, it is proven that the solution of a heat conductivity problem in the statement in which it is given in this section is unique, i.e., there are no two independent solutions which could satisfy differential equation (1-17), initial condition (1-18), boundary condition (1-19) and, possibly, the conjugation conditions (if the body is piecewise-homogeneous). Solutions may differ only in their form of presentation; they all yield a unique, unambiguously defined temperature field.

Up to now, we have spoken of the so-called classical solution of the problem of heat conductivity, i.e., a continuous function of coordinates and time assigned in a certain area, having continuous derivatives up to second order inclusively with respect to the spatial variables and up to first order with respect to the time variable, converting the corresponding differential equation to an identity and satisfying the edge conditions.

The requirement to seek a classical solution to the heat conductivity problem places very rigid limitations on the assigned functions included in the differential equation and edge conditions.

In practice, there is usually little concern as to whether a series (or integral) representing the solution of a problem can be twice differentiated with respect to the coordinates and once with respect to time, but rather the only requirement is that of continuity of the solution. We note that this approach is physically quite satisfactory.

The solution represented by an evenly converging series (or integral), the sum of which does not have a sufficient quantity of partial derivatives, is called the generalized solution.

Let us explain this.

Suppose $\{T_n\}$ is an infinite sequence of solutions. Then the series

$$\sum_{n=1}^{\infty} T_n \tag{1-27}$$

regardless of its convergence is called the formal solution.

If T_n are the classical solutions, series (1-27) converges evenly and its sum has the necessary partial derivatives, then this sum is the classical solution; if the condition of even convergence of the series is fulfilled, but its sum does not have a sufficient number of partial derivatives, this solution is a general solution. The situation is similar for the case of representation of the solution in the form of an integral.

The concept of the generalized solution was introduced by S. L. Sobolev [115]; we present here its simple definition.

The generalized solution depends uniquely and continuously on the edge condition; it is therefore correct.

In this book, instead of attempting to find the classical solutions to the problem of heat conductivity, we will be primarily occupied in finding generalized solutions. In accordance with this approach, it is sufficient to require that the assigned functions p, Q, f, ϕ, ψ, η, included in the differential equation and edge conditions, be piecewise-smooth functions of their own arguments[1]. This requirement is always fulfilled for the problems of heat conductivity which will be analyzed in this book.

[1]Function $u(\xi)$ is called piecewise-smooth over interval $[R_1, R_2]$ if the function $u(\xi)$ itself and its first derivative $u'(\xi)$ are piecewise-continuous over this interval; the function $u(\xi)$ is called piecewise-continuous over interval $[R_1, R_2]$ if it is continuous everywhere with the exception of a finite number of points of finite discontinuity.

Therefore, the problems of existence and uniqueness of the solution will not be discussed further. In seeking out the temperature function by various methods, we will assume that the assigned functions satisfy all conditions which are necessary for realization of these methods.

CHAPTER 2. INITIAL DATA FOR CALCULATION OF TEMPERATURE FIELDS IN CONCRETE WATER ENGINEERING STRUCTURES

2-1. Heat-Physical Characteristics of Concrete Used in Dams and of Rock Formations

Water engineering concrete is a complex polydispersed heterogeneous polymineral system, consisting of solid crystalline, polycrystalline and amorphous particles, water and a gas phase.

The precise mathematical description of the process of heat transfer in such systems involves significant difficulties [136, 137]. Therefore, in calculating the temperature fields of concrete bodies, we generally use a phenomenological theory of heat conductivity, the basic elements of which were presented in Chapter 1.

The phenomenological approach to the investigation of heat conductivity in concrete structures during the period of construction and use is used in the present book. The concrete is looked upon as a quasihomogeneous isotropic[1] material with heat-physical characteristics averaged within the limits of physically rather large volumes.

The practice of many theoretical calculation and experimental studies performed under laboratory and natural conditions has shown that this structural model of concrete in combination with the phenomenological theory of heat conductivity is quite suitable for engineering applications.

The heat-physical characteristics of the material are the heat conductivity factor λ, temperature conductivity factor a and specific volumetric heat capacity $c\gamma$ (product of specific heat capacity c and density γ).

The heat conductivity factor λ characterizes the intensity of the process of heat conductivity in the substance and is numerically equal to the heat flux density resulting from heat conductivity with a temperature gradient of unity. The units of measurement of λ are: engineering system -- kcal/(m·hr·C); SI system -- w/(m·C); conversion formula 1 kcal/(m·hr·C) = = 1.1630 w/(m·C).

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[1] Generally, processes of sedimentation resulting from differences in the density of the components of a concrete mixture occur before it sets. Sedimentation may generally lead to different properties of the concrete in different directions. However, up to now this phenomenon has been little studied. Furthermore, it must be expected on the basis of general conditions that the sedimentation possible in a concrete mixture would have little influence on the heat-physical properties of the concrete.

The specific heat capacity c is numerically equal to the quantity of heat necessary to increase the temperature of a unit mass of matter by 1 Celcius. The units of measurement of c are: engineering system -- kcal/(kg·C), SI -- J/(kg·C), conversion formula -- 1 kcal/(kg·C) = $4.1868 \cdot 10^3$ J/(kg·C).

The density γ is numerically equal to the mass per unit volume of the substance. The units of measurement of γ are: engineering system -- kg/m^3, SI -- kg/m^3.

The temperature conductivity factor a is numerically equal to the ratio of the heat conductivity factor λ to the volumetric specific heat capacity $c\gamma$ of the substance ($a = \lambda/c\gamma$). The units of measurement of a are: engineering system -- m^2/hr, SI -- m^2/s, conversion formula -- 1 m^2/hr = $2.7773 \cdot 10^{-4}$ m^2/s.

The heat-physical characteristics of concrete depend on many factors: the type and consumption of cement and filler, water/cement ratio, conditions of drying and setting of the concrete, its age, temperature, etc.

One of the most important factors, which goes far toward determining the heat-physical characteristics of the concrete, is the type of filler used. Fillers in concrete in most cases do not enter into the chemical reaction between the binder and the water; they form the rigid skeleton of the material. The coarse filler (particle size 5-150 mm, in rare cases up to 300-400 mm) is present in concrete as gravel or rubble made of compact rock (granite, diabase, limestone, gabbro, dolomite, etc.), its content being from 1200 kg/m^3 to 1800 kg/m^3. The fine filler (particle size up to 5 mm) is natural or artificial sand, its content being 500-800 kg/m^3.

The highest values of heat conductivity and temperature conductivity factor are those of concrete made with quartz filler, the lowest -- concrete made with basalt or rhyolite filler. This can be seen, for example, from the data produced by American researchers [157] (Table 2-1).

TABLE 2-1. TEMPERATURE CONDUCTIVITY FACTOR OF CONCRETES MADE WITH VARIOUS COARSE FILLERS

Coarse Filler	a, m^2/hr	Coarse Filler	a, m^2/hr
Quartzite	0.0054	Granite	0.0040
Limestone	0.0047	Rhyolite	0.0033
Dolomite	0.0046	Basalt	0.0030

The influence of the type of cement, size of filler and water/cement ratio is less significant.

In water engineering concretes, Portland cement (PC) and its varieties such as Portland cement with moderate exothermy, slag-Portland cement [SPC], puzzolan Portland cement [PPC], etc. are used. The cement content in concrete varies from 100 to 400 kg/m^3.

According to American data [173], the heat conductivity factor of cement stone based on various Portland cements falls between 0.75 and 0.8 kcal/(m·hr·C). Similar values of heat-physical characteristics are produced by French (λ = 0.85 kcal/(m·hr·C) [162]) and Soviet (a = 0.001 m^2/hr [28]) cements.

An increase in filler diameter by 50% increases λ of concrete by only 5% [162, 173].

The water in concrete amounts of 100-200 kg/m^3, the water/cement ratio W/C (ratio of mass of water to mass of cement, considering only the free water, not absorbed by the filler) is 0.4-0.8. Variation in W/C from 0.4 to 0.8 decreases λ by 5-10% [2, 162, 173]. An increase in the drying rate from the natural rate to that used by drying with heating also changes heat conductivity of concrete very little (λ decreases by 5% [162]).

Water engineering concretes are heavy concretes, their density varying from 2300 to 2600 kg/m^3. The porosity of the concrete is 6-12%.

The heat conductivity of concrete as a function of its porosity is satisfactorily described by the theory of generalized conductivity of V. I. Odelevskiy [83, 91, 137]. According to this theory, the heat conductivity factor of a two-component matrix "solid + air" system λ_c, on the assumption that the heat conductivity of air is negligible, is determined by the formula

$$\lambda_c = \lambda_M \frac{1-p}{2+p}, \tag{2-1}$$

where λ_M is the heat conductivity factor of the matrix element (in this case the solid skeleton of the concrete); p is the porosity as a fraction of the whole.

The selection of a type of filler and determination of the composition of the concrete are primarily based on considerations not related to heat-physical characteristics (distance from the construction site to quarries containing suitable fillers providing the required strength, impermeability and other physical-mechanical properties of the concrete). Therefore, the heat-physical characteristics of the concrete must be taken as given.

The heat-physical characteristics of concrete are usually determined by experimental methods either under laboratory conditions using specimens of production concrete or concrete of similar composition [1, 4, 28, 37], or under natural conditions directly at the construction site [5, 47]. Methods involving stable and unstable heat fluxes are used. In the latter case, the methods most commonly used are those based on the regularities of a regular quasistable mode [5], as well as probe methods [47].

TABLE 2-2. HEAT-PHYSICAL CHARACTERISTICS OF THE CONCRETE OF CERTAIN DAMS

No.	Name of Dam, Country	Type of Filler	Heat-Physical Characteristics			
			γ, kg/m^3	λ, kcal/(m·hr·C)	c, (kcal/(kg·C))	a, m^2/hr
1	Hoover Dam, USA	Limestone, granite	2500	2.50	0.218	0.0046
2	Grand Coulee, USA	Basalt,	2534	1.61	0.225	0.0028
3	Frayant	Quartzite, granite, rhyolite	2465	1.83	0.223	0.0033
4	Shasta, USA	Andesite, shale	2510	1.95	0.226	0.0034
5	Anchor, USA	Andesite, limestone	2388	1.70	0.235	0.0030
6	Flaming Gorge, USA	Limestone, sandstone	2411	2.63	0.227	0.0048
7	Detroit, USA	--	2403	2.38	0.248	0.0040
8	Pieve di Cadore, Italy	Limestone	--	2.64	--	--
9	Morasco, Italy	--	--	3.00	--	--
10	Dieksans, Switzerland	--	--	1.50	--	--
11	Tonoyama, Japan	Sandstone	2350	1.92	0.210	0.0039
12	Kamishiba, Japan	Sandstone	2378	1.72	0.230	0.0032
13	Mamakanskiy, USSR	--	--	2.13	--	0.0038
14	Bratsk, USSR	Granite, diabase	2450	2.10	0.230	0.0037
15	Krasnoyarsk, USSR	Granite	2400	1.94	0.230	0.0035
16	Toktolgul'sk, USSR	Granite, diabase	2520	2.14	0.235	0.0036

Note: The data presented in [157] on American dams (Nos. 1-6) were averaged for the 10-40 C temperature interval.

In formulating the problem of heat conductivity, the nature of the dependence of heat-physical characteristics of the concrete on temperature and time (age) is quite important.

Studies performed by various authors [2, 157, 162, 173]have shown that up to 40-50 C the temperature has little influence on the heat conductivity and temperature conductivity of the concretes. According to the American Institute of Concrete [157], in this temperature interval the heat conductivity coefficient of concrete changes by not over 4%, the temperature conductivity coefficient -- by 10%. The heat-physical characteristics of concrete change but slightly with age (by not over 5-7% for concrete between 3 and 180 days of age [2, 162, 173]).

Table 2-2 presents values of the heat-physical characteristics of concretes used in the construction of a number of dams.

American researchers recommend the following formulas for preliminary estimates of the heat-physical characteristics of water engineering concrete λ_b, kcal/(m·hr·C) and c_b, kcal/(kg·C):

$$\lambda_b = \sum_{i=1}^{n} G_i f_{1i};$$
$$c_b = \sum_{i=1}^{n} G_i f_{2i}. \qquad (2\text{-}2)$$

where G_i is the percent content (by mass) of component i in the concrete mixture; f_{1i} and f_{2i} are parameters. The numerical values of f_{1i} and f_{2i} for concrete based on various fillers were established in the course of extensive combined investigations in the construction of the Hoover Dam (Boulder Canyon Project) [173]. These values, averaged for the 21-43 C temperature interval, are presented in Table 2-3.

TABLE 2-3. MEAN VALUES OF PARAMETERS f_{1i} AND f_{2i} [see formula (2-2)]

Component	f_{1i}	f_{2i}	Component	f_{1i}	f_{2i}
Water	0.00515	0.0100	Basalt	0.0164	0.00184
Cement	0.0110	0.00134	Dolomite	0.0364	0.00197
Quartz sand	0.0265	0.00178	Granite	0.0253	0.00172
Rhyolite	0.0162	0.00186	Limestone	0.0339	0.00182
			Quartzite	0.0400	0.00173

Calculation example. During the construction of the Grand Coulee Dam (USA), a concrete mixture of the following composition was used [157]: cement -- 224, sand -- 582, coarse filler -- basalt -- 1523, water -- 134 kg/m^3. The density of the concrete γ_b = 2463 kg/m^3. The results of calculation of λ_b and c_b based on formula (2-2) and the data of Table 2-3 are presented below in tabular form.

The calculated values were λ_b = 1.77 kcal/(m·hr·C), c_b = 0.222 kcal/(kg·C), $a_b = \lambda_b/c_b\gamma_b$ = 0.0031 m^2/hr. The values experimentally determined (Table 2-2) were: λ_b = 1.61 kcal/(m·hr·C), c_b = 0.225 kcal/(kg·C), a_b = 0.0028 m^2/hr. The variation is about 10%, which can be considered satisfactory for the first approximation.

Component	Consump. Mater.		λ_b		c_b	
	kg/m^3	G_i, mass %	f_{1i}	$G_i f_{1i}$	f_{2i}	$G_i f_{2i}$
Water	134	5,4	0,00515	0,028	0,0100	0,540
Cement	224	9,1	0,01100	0,100	0,00134	0,0122
Sand	582	23,6	0,02650	0,625	0,00178	0,0420
Rubble (basalt)	1 523	61,9	0,01640	1,015	0,00184	0,1136
	2 463	100		1,77		0,222

High concrete dams are constructed on bases consisting of dead rock, igneous metamorphic (diabase, basalt, granite, gabbro) or sedimentary (limestone, dolomite, sandstone). The rock bases are usually split by various systems of joints, the degree of jointing varying quite broadly for all types of rock. This results in large intervals of values of physical and mechanical characteristics of bases of any given type [109]. However, the intervals of values of heat-physical characteristics of such bases are narrower.

In the literature, we find very little information on the heat-physical characteristics of rock in lump form or of bases of water engineering structures with the rock in its natural deposition. Therefore, we present below in Table 2-4 some data [172] which need further refinement for each specific case.

TABLE 2-4. HEAT-PHYSICAL CHARACTERISTICS OF ROCKS USED AS CONCRETE FILLERS

Rock	γ, kg/m^3	λ, kcal/(m·hr·C)	c, kcal/(kg·C)	a, m^2/hr
Quartzite	2 430	3,04	0,217	0,0053
Dolomite	2 510	2,86	0,231	0,0049
Limestone	2 450	2,74	0,224	0,0050
Granite	2 410	2,24	0,220	0,0042
Basalt	2 530	1,81	0,226	0,0032
Rhyolite	2 350	1,78	0,226	0,0034

To achieve this refinement, considering the porosity of the material, we can use formula (2-1). In this formula, λ_M should be taken to mean the heat conductivity factor of the solid skeleton of the base, and determination of porosity p should be performed considering that portion of the porosity resulting from jointing of the base.

2-2. Heat Liberation of Concrete

As concrete cures, a significant quantity of heat is liberated, as a result primarily of the exothermic reactions of hydration of cement.

The heat liberation of concrete depends on the chemical and mineralogical composition of the cement, its consumption and fineness, the water/cement ratio, the presence of puzzolan, the age, temperature during curing and other factors.

The primary components of Portland cement used in water engineering construction are bicalcium silicate $2CaO \cdot SiO_2 (C_2S)$[1], tricalcium silicate $3CaO \cdot SiO_2 (C_3S)$, tricalcium aluminate $3CaO \cdot Al_2O_3 (C_3A)$, and tetracalcium aluminoferrate $4CaO \cdot Al_2O_3 \cdot F_2O_3 (C_4AF)$.

The data of V. A. Kind, S. D. Okorokov and S. L. Vol'fson [56] on heat liberation of these components are presented in Table 2-5.

TABLE 2-5. HEAT LIBERATION OF THE PRIMARY MINERALS OF PORTLAND CEMENT

Mineral	Quantity of heat liberated, kcal/kg, at various curing times, days				
	3	7	28	90	180
C_2S	97	110	116	124	135
C_3S	15	25	40	47	55
C_3A	141	158	209	222	245
C_4AF	42	60	90	99	—

As we can easily see, the minerals C_3A and C_3S are high thermic minerals, with significant heat liberation intensities during the initial period of curing, whereas C_4AF and particularly C_2S are low thermic, with low early heat liberation.

Upon curing of Portland cement, the products of hydration of some minerals may influence the hydration of others, so that the total thermal effect generally does not follow the additive rule.

Concretes based on cements of varying mineralogical composition may differ significantly in heat liberation.

The specific heat liberation in concrete, i.e., the heat liberation per unit mass of cement, depends within relatively narrow limits on the water/cement ratio. Thus, in experiments by Ts. G. Ginzburg and L. I. Kots [31], the specific heat liberation for W/C of 0.43 to 0.76 at 7 days age increased for pure clinker Portland cement and slag Portland cement by an average of 8%. The specific heat liberation Q_{sp}, kcal/kg, varies nearly linearly with W/C [50]

[1] Parentheses show standard symbols.

$$Q_{sp} = Q_{0sp}(1 + \alpha \frac{B}{C}),$$

where α is a coefficient which for plastic concrete mixtures with cone slump OK = 2-12 cm can be taken as

$$\alpha \simeq (\frac{1}{6} - \frac{1}{8}) \text{ OK (where } \tau \geq \text{ 5-7 days).}$$

Finer grinding accelerates the process of hydration of the cement and, consequently, intensifies heat liberation; the influence of the fineness of the cement is felt only during the early period of curing. According to the data of earlier research [31], an increase in specific surface of cement by 100 cm^2/g causes an increase in heat liberation as follows, on the average: at 1 day age by 3.2 kcal/kg, at 28 days age by 1.8 kcal/kg, at 90 days age by 1.5 kcal/kg, at 1 year age by 0.7 kcal/kg.

At the cement consumptions used in water engineering concrete (100-400 kg/m^3), heat liberation depends linearly on cement content; an increase in the consumption of cement, for example from 150 to 300 kg/m^3, increases heat liberation only approximately by 1.5 times [50].

Most puzzolans help to reduce heat liberation of concrete, the relative value of this reduction being significantly less than the percentage of this additive introduced. For a rough estimate of the influence of puzzolan on heat liberation, it is recommended by [157, 172] that one assume that puzzolan liberates 50% of the heat which would be liberated by the cement replaced by the puzzolan.

Table 2-6 presents data on the heat liberation of concretes used in the construction of several domestic dams.

The heat liberation of concrete is generally established by experimental investigation of material of the production composition.

If experimental data are not available, for approximate preliminary calculations the heat liberation of the concrete can be determined by calculating the heat of hydration of the cement using the formula recommended by GOST 4798-69 [13]

$$Q_n = aC_3S + bC_2S + cC_3A + dC_4AF, \quad (2\text{-}3)$$

where Q_n is the heat of hydration of the cement after n days of curing, kcal/kg; a, b, c, d are coefficients taken from the data of Table 2-7; C_3S, C_2S, C_3A and C_4AF represent the content of the corresponding cement minerals, %.

TABLE 2-6. HEAT LIBERATION OF CONCRETES OF CERTAIN DOMESTIC DAMS

1 Плотина	2 Тип цемента	3 Расход цемента, на 1 м³ бетона, кг	4 В/Ц	5 Начальная температура бетонной смеси, °C	6 Адиабатическое тепловыделение, ккал/кг, в разные сроки твердения, сутки				
					1	3	7	14	28
7 Братская	8 ШПЦ	240	0,55	8	12,4	24,8	48,9	65,1	73,6
	ШПЦ	170	0,75	23	18,4	42,1	67,6	78,8	84,2
	ПЦ	230	0,50	7	15,0	45,4	80,4	92,2	95,6
	ПЦ	250	0,50	14	22,6	50,9	70,3	78,0	79,6
9 Красноярская	10 ШПЦ	250	0,52	20—23	33	59	77	82	—
	ПЦ	250	0,50	20—23	55	78	90	100	—
11 Токтогульская	12 ПЦ	220	0,50	15	21,2	52,0	62,4	65,2	67,8

Key: 1, Dam; 2, Type of Cement; 3, Cement Consumption per m^3 Concrete, kg; 4, W/C; 5, Initial Temperature of Concrete Mixture, C; 6, Adiabatic Heat Liberation, kcal/kg, at Various Curing Times, Days; 7, Bratsk; 8, SPC, SPC, PCP, PCP; 9, Krasnoyarsk; 10, SPC, PC; 11, Toktogul'sk; 12, PC

TABLE 2-7. VALUES OF COEFFICIENTS, kcal/kg, IN FORMULA (2-3)

Curing Time, days	*a*	*b*	*c*	*d*
3	0,929	0,159	1,517	−0,119
7	1,093	0,231	2,069	−0,414
28	1,142	0,153	2,299	+0,140
90	1,183	0,231	2,458	+0,332
180	1,220	0,445	2,457	+0,382
365	1,269	0,532	2,525	+0,400

Table 2-8 presents approximate averaged values of heat liberation of cements.

TABLE 2-8. HEAT LIBERATION OF CERTAIN CEMENTS

Cement Type	Cement Grade	Isothermal (T_0=15 C) heat lib., kcal/kg, at Various Curing Times, days		
		3	7	28
Portland cement	600	75	90	100
	500	65	75	85
	400	55	65	75
	300	45	55	60
Puzzolan Port cement	400	30	45	65
	300	25	40	55
Slag-Portland cement	300	25	45	60

The heat liberation in concrete is a result of complex physical and chemical processes occurring upon curing of the cement stone. The rate of these processes depends both on time and on temperature. Consequently, heat liberation is also a function of time and temperature.

As we can see from the data presented above, as well as that known from experimental studies, during the first 7 to 10 days of curing some 70 to 80% of the total heat of hydration is liberated; subsequently, the process of heat liberation is significantly retarded.

The following formula is frequently used to consider exothermy as a function of time alone [18, 156]

$$Q = Q_{max}(1 - e^{-m\tau}), \tag{2-4}$$

where Q_{max} is the maximum quantity of heat which can be liberated in the concrete of the given composition with full hydration of the cement, in other words the total heat liberation of the concrete; m is a parameter which depends on the type of cement; for concretes based on Portland cement, it varies between 0.010 and 0.015 1/hr.

This formula was produced on the assumption that the intensity of heat liberation q is proportional to the quantity of heat not yet liberated at any given moment in time

$$q = \frac{dQ}{d\tau} = m(Q_{max} - Q). \tag{2-5}$$

Integration of equation (2-5) with the initial condition

$$Q \mid_{\tau=0} = 0$$

leads to the heat liberation function (2-4), and, consequently, to the heat liberation intensity function

$$q = q_0 e^{-m?}. \tag{2-6}$$

Comparison of experimental data with calculation data shows that the heat liberation intensity function (2-6) only very approximately describes the heat liberation curve.

However, we can suggest certain generalizations of formula (2-4), which allow us to approximate the heat liberation curve of concrete of any composition with good accuracy.

One of these generalizations is based on the assumption that the law of proportionality (2-5) is correct for brief intervals of time and the heat liberation intensity function can be fixed in the form

$$q = q_\nu e^{-m_\nu \tau} \tag{2-7}$$

where q_ν and m_ν are parameters, piecewise-constant functions of time, defined in $(\tau_{\nu-1}, \tau_\nu)$, i.e., are equal to

$$q_1, m_1, \text{ where } 0 < \tau < \tau_1$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$q_i, m_i \text{ where } \tau_{i-1} < \tau < \tau$$

Parameters q_ν and m_ν, as well as the length of time sectors $(\tau_{\nu-1}, \tau_\nu)$ within which q_ν and m_ν are assumed constant, are established from experimental data on the heat liberation of concretes.

Let us assume that the curve in Figure 2-1 is the experimental curve of heat liberation in a concrete.

Let us discuss the time interval in question over a number of time sectors $(\tau_{\nu-1}, \tau_\nu)$ $(\nu = 1, 2, \ldots, i)$. Each of these sectors, in turn, will be divided into two equal intervals of length $\Delta\tau$, so that

$$\tau_\nu - \tau_{\nu-1} = 2\Delta\tau.$$

As we can easily see, the heat liberation during time sector $(\tau_{\nu-1}, \tau_\nu)$ is

$$\Delta Q^{(\nu)} = Q^{(\nu)} - Q^{(\nu-1)} =$$
$$= Q^{(\nu)}_{\max} \left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu (\tau_{\nu-1} + 2\Delta\tau)}\right),$$

where

$$Q^{(\nu)}_{\max} = \frac{q_\nu}{m_\nu}.$$

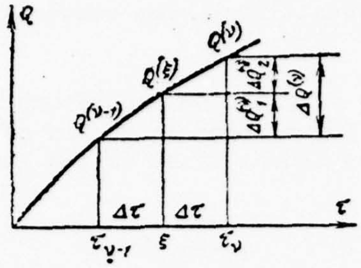

Figure 2-1. Diagram of Subdivision of Experimental Curve of Heat Liberation into Sectors of Approximation of Exponential Intervals

Similarly, we can write:

$$\Delta Q_1^{(\nu)} = Q^{(\xi)} - Q^{(\nu-1)} = Q^{(\nu)}_{\max}\left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu(\tau_{\nu-1}+\Delta\tau)}\right).$$

Taking the ratio $\Delta Q^{(\nu)}/\Delta Q_1^{(\nu)}$, we produce:

$$\frac{\Delta Q^{(\nu)}}{\Delta Q_1^{(\nu)}} = \frac{1-\exp(-m_\nu 2\Delta\tau)}{1-\exp(-m_\nu \Delta\tau)} = 1 + e^{-m_\nu \Delta\tau}.$$

From which

$$m_\nu = \frac{1}{\Delta\tau_\nu}\ln\frac{\Delta Q_1^{(\nu)}}{\Delta Q_2^{(\nu)}}; \qquad (2\text{-}8)$$

$$q_\nu = \frac{m_\nu\,\Delta Q^{(\nu)}}{\exp(-m_\nu \tau_{\nu-1}) - \exp(-m_\nu \tau_\nu)}, \qquad (2\text{-}9)$$

where $\Delta Q_2^{(\nu)} = Q^{(\nu)} - Q^{(\xi)}$ (see Figure 2-1).

Formulas (2-8) and (2-9) are the basic formulas for determination of the parameters q_ν and m_ν.

In processing experimental data on the heat liberation of concretes curing under adiabatic conditions, we should use the formula

$$\Delta Q = c\gamma\Delta T,$$

where ΔT is the adiabatic temperature rise.

In the general case the approximation curve thus constructed will not be smooth, and breaks may occur at the ends of the time sectors. Analysis, however, has shown that these effects are slight and have little influence on the results of calculation of the temperature in the body.

However, we can slightly alter the method proposed and "smooth" the curve: the experimental curve is divided into time sectors, each of which partially overlaps the preceding sector, and the values of q_ν and m_ν defined for the entire time sectors are assigned to its "free" portion.

In practice, this altered method is quite simple to realize as follows.

The curve of heat liberation is divided into equal time sectors of length $\Delta\tau$, so that

$$\tau_\nu - \tau_{\nu-1} = \Delta\tau,$$

and the calculation of m_ν and q_ν is based on the formulas

$$m_\nu = \frac{1}{\Delta\tau} \ln \frac{\Delta Q^{(\nu-1)}}{\Delta Q^{(\nu)}}; \tag{2-10}$$

$$q_\nu = \frac{m_\nu (\Delta Q^{(\nu-1)} + \Delta Q^{(\nu)})}{\exp(-m_\nu \tau_{\nu-2}) - \exp(-m_\nu \tau_\nu)}. \tag{2-11}$$

In summarizing formula (2-6), we can also utilize the assumption that

$$q = \sum_{\nu=1}^{i} q_\nu e^{-\nu m\tau}, \tag{2-12}$$

where q_ν and m are parameters.

The heat liberation function in this case is:

$$Q = \sum_{\nu=1}^{i} \frac{q_\nu}{\nu m} (1 - e^{-\nu m\tau}). \tag{2-13}$$

Parameters q_ν and m_ν, as well as the number of the exponents in formula (2-13) i are determined from the condition of satisfactory approximation to the experimental curve of heat liberation.

The following method of solution of the problem is possible.

Suppose n experimental points are fixed on the "heat liberation-time" curve. Let us assume at first i = 1, i.e., assume

$$Q = \frac{q_1}{m}(1 - e^{-m\tau}),$$

and use the method of least squares to define q_1 and m. The value of parameter m produced will be used in the future as an assigned parameter. Consequently, to define the i parameters q_ν we utilize the system

$$Q_k = \sum_{\nu=1}^{i} \frac{q_\nu}{\nu m}(1 - e^{-\nu m \tau_k})\ (k = 1, 2, \ldots, n,\ n > i). \tag{2-14}$$

Let us supplement system (2-14) with the arbitrary equation

$$Q_{max} = \sum_{\nu=1}^{i} \frac{q_\nu}{\nu m}(1 - e^{-\nu m \tau_{max}}), \tag{2-15}$$

which follows from the requirement of precise equality of the summary heat liberation Q_{max} calculated by formula (2-13) to the assigned heat liberation at moment τ_{max}.

The incompatible system (2-14) with arbitrary equation (2-15) is solved by the method of least squares. The selection of the number of exponents i is established by tests.

Analysis of the experimental data has shown that formulas such as (2-7) and (2-12) satisfactorily describe the dependence of exothermy of cements and water engineering concretes on time.

The empirical formulas earlier suggested by various authors are partial in nature and cannot be recommended for approximation of any "heat liberation-time" curve.

The most popular methods for laboratory determination of exothermy of concrete are: 1) the method of dissolution, 2) the method of the isothermal calorimeter (thermos method), 3) the adiabatic calorimeter method. Using the first two methods, testing of concrete for heat liberation is performed at a constant or near constant temperature (isothermal curing conditions).

In the third method, the specimen is heat insulated, all of the heat liberated goes to heat the specimen, so that the curing of the concrete occurs under conditions of continually changing (increasing) temperature.

In a study of the dependence of heat liberation on temperature, various curing modes are achieved by selecting various constant specimen temperatures (isothermal conditions) or by selecting various initial temperatures of the concrete mixture (adiabatic conditions).

Some of the results of such tests are presented in Table 2-9.

TABLE 2-9. HEAT LIBERATION OF CONCRETES CURING AT VARIOUS TEMPERATURES

1 Тип цемента	2 Расход цемента на 1 м³ бетона, кг	3 В/Ц	4 Начальная температура бетонной смеси, °C	5 Удельное тепловыделение, ккал/кг, в разные сроки твердения, сутки: 1	3	7	6 Литература
7	8 Изотермические условия твердения при различных постоянных температурах образца						
Портландцемент	500	0,5	5 10 22	17 22 23	22 26 39	32 37 45	[118]
9 Шлакопортландцемент	500	0,5	5 10 22	15 20 21	26 28 27	35 39 40	[118]
	10 Адиабатические условия твердения при различных начальных температурах бетонной смеси						
11 Портландцемент	250	0,58	18 28	21 30	42 62	54 81	[29]
12 Пуццолановый портландцемент	320	0,61	19 27	11 15	25 34	21 54	13 То же

Key: 1, Type of Cement; 2, Cement Consumption per m^3 of Concrete, kg; 3, W/C; 4, Initial Temperature of Concrete Mixture, C; 5, Specific Heat Liberation, kcal/kg, at Various Curing Times, Days; 6, Reference; 7, Portland Cement; 8, Isothermal Curing Conditions with Various Constant Specimen Temperatures; 9, Slag-Portland Cement; 10, Adiabatic Curing Conditions with Various Initial Temperatures of Concrete Mixture; 11, Portland Cement; 12, Puzzolan Portland Cement; 13, [29]

An increase in the curing temperature generally leads to more intensive heat liberation. Cases are observed in which even though the initial temperature of the concrete mixture is lower, the summary heat liberation effect is higher (Figure 2-2).

A. A. Gvosdev [23] considers the dependence of heat liberation in concrete on temperature, summarizing the proportionality rule (2-5) and writes it as

$$q=\frac{dQ}{d\tau}=(Q_{max}-Q)f(T), \tag{2-16}$$

where f(T) is a certain function of temperature.

From this, considering the initial condition

$$Q\mid_{\tau=0}=0$$

it follows that

$$Q=Q_{max}\left[1-\exp\left(-\int_0^\tau f(T)\,d\tau\right)\right]$$

and (2-17)

$$q=Q_{max}f(T)\exp\left(-\int_0^\tau f(T)\,d\tau\right).$$

S. V. Aleksandrovskiy [2], developing a suggestion by A. A. Gvozdov, accepts

$$f(T) = BT$$

and for the intensity function of heat liberation, produces the expression

$$q=\omega(\tau)T;$$

$$\omega(\tau)=\frac{BT_{\text{пр}}(T_{\text{пр}}-T_0)}{T_{\text{пр}}+T_0\left[\exp(BT_{\text{пр}}\tau)-1\right]}, \tag{2-18}$$

where B is the adiabatic heat liberation rate parameter of the cement in the concrete of predetermined composition; T_m is the maximum temperature of adiabatic solidification of the concrete; T_0 is the initial temperature of the concrete.

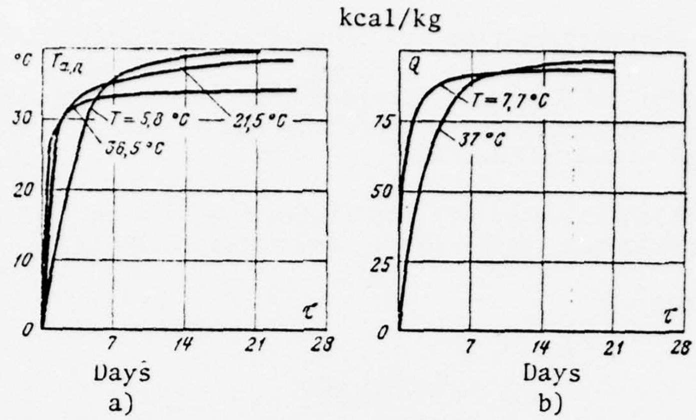

Figure 2-2. Experimental Data on Heat Liberation of Water Engineering Concrete Curing Under Adiabatic Conditions with Various Initial Temperatures of the Concrete Mixture. a, Data from Central Research Laboratory (Japan) [171]; b, Data from Bureau of Reclamation (USA) [172]

P. I. Vasil'yev and M. A. Zubritskaya [17] present the heat liberation intensity function in a more general form than A. A. Grovdev. They believe that the heat liberation rate in concrete of a fixed composition for a given moment in time at a given point is a certain function of temperature T and the quantity of heat liberated Q:

$$q = F(T, Q) = F[T(x, y, z, \tau);\ \ Q(x, y, z, \tau)];$$
$$Q = \int_{\tau_0}^{\tau} F(T, Q)\, d\tau. \qquad (2\text{-}19)$$

Here

$$Q\,|_{\tau=\tau_0} = 0, \quad Q_{max} = \int_{\tau_0}^{\infty} F(T, Q)\, d\tau.$$

The function of A. A. Gvozdev f(T) is similar in its content to the temperature-time function $f_a(T)$ analyzed by Rastrup [165] and others, while $\int_0^\tau f(T)d\tau$ is similar to the adjusted time

$$\tau_{пр} = \int_0^{\tau} f_a(T)\, d\tau. \qquad (2\text{-}20)$$

The sense of the temperature-time function is as follows.

Assume the curing of two concrete specimens of the same composition occurs at different but time-constant temperatures: the standard T_a in the first case and arbitrary T in the second.

Suppose the time which is necessary for liberation of a certain quantity of heat Q in each specimen is equal to τ_{ct} and τ respectively.

It is affirmed that

$$\tau_{\text{ст}} = f_a(T)\tau. \qquad (2\text{-}21)$$

If temperature T in the process of curing changes, then

$$\tau_{\text{ст}} = \int_0^{\tau} f_a(T)\, d\tau. \qquad (2\text{-}22)$$

In determining the form of function $f_a(T)$, Rastrup [165] takes as his basis the known law of chemistry according to which the rate of a chemical reaction doubles when the temperature is elevated by 10 C, and proposes for the temperature-time function the expression

$$f_a = 2^{\frac{T-T_a}{10}}. \qquad (2\text{-}23)$$

Other expressions are also known for the temperature-time function.

L. Mejzlik [161] in calculating the temperature fields of concrete structures, uses the following heat liberation function

$$Q = Q_{max}\left[1 - \exp\left(-m\int_0^{\tau} f_a(T)\, d\tau\right)\right].$$

The parameters Q_{max} and m are determined from the curve of heat liberation of concrete at a constant temperature of 15 C. The temperature-time function used is different for different temperature intervals.

G. I. Chilingarishvili [134, 135] suggests a heat liberation function of the following form

$$q = q_{max} \exp(-m\tau_m^{-\beta}),$$

containing three parameters Q_{max}, m and β, the numerical values of which are established on the basis of assigned temperatures to which the concrete is heated in three different periods of time.

The theory of G. D. Vishnevetskiy [20, 21] is based on the concept that the molecular transfer of moisture from macropores into the zone of hydration through the microporous skeleton of the cement stone has primary influence on processes occurring in the curing concrete.

For the heat liberation intensity function, he produces the expression

$$q = \frac{Q_{max}}{\tau_{max}} K'(\tau_m),$$

where Q_{max} is the maximum heat liberation of the concrete over the entire period of continuing hydration

$$K'(\tau_{up}) = \frac{3}{2\sqrt{\tau_{up}}}(1 - \sqrt{\tau_{up}})^2;$$

$$\tau_{np} = \int_0^{\tau} \frac{1}{\tau_{max}} d\tau,$$

τ_{max} is a certain limiting age, in days, determined from the empirical formula

$$\tau_{max} = 1000 \exp(-0.06T).$$

In the opinion of G. D. Vishnevetskiy, the formula of Rastrup (2-23) where T > 25 C yields elevated values, since at these temperatures the fact limiting the rate of hydration is not the rate of the chemical reactions, but rather the rate of effusion delivery of water to the hydration zone.

I. D. Zaporozhets [48, 50] applies the rule of affective masses[1] to the process of hydration of cement in the curing concrete, writing it in the following form:

$$\frac{dB_x}{d\tau} = k(T)(B_{A_0} - B_x)^m, \tag{2-24}$$

where B_x is the quantity of chemically bonded water in a unit volume of concrete at moment in time τ; B_{A_0} is the maximum quantity of water which can participate in the process of hydration (according to the terminology of I. D. Zaporozhets, the initial reserve of active water); m is the order of the reaction, $m \neq 1$; k(T) is the reaction rate constant.

Integrating expression (2-24) with the initial condition

$$B_x \mid_{\tau=0} = 0$$

and assuming that the values of hydration and heat liberation are proportional, i.e.,

$$Q = hB_x,$$

where h is the proportionality coefficient, equal to the quantity of heat liberated upon bonding of a unit mass of water, the author produces the heat liberation function

$$Q = hB_{A_0}\left\{1 - \left[1 + (m-1)\,B_{A_0}^{m-1}\int_0^{\tau} k(T)\,d\tau\right]^{-\frac{1}{m-1}}\right\} \tag{2-25}$$

and the concrete heat liberation intensity function

$$q = \frac{dQ}{d\tau} = hB_{A_0}^{m}\,k(T)\left[1 + (m-1)\,B_{A_0}^{m-1}\int_0^{\tau} k(T)\,d\tau\right]^{-\frac{m}{m-1}}. \tag{2-26}$$

---

[1]The law of effective masses establishes that for simple chemical reactions at a given temperature, the rate of the reaction is proportional to the concentration of reagents in powers equal to the stoichiometric coefficients of the reaction.

It is assumed that parameters B_{A_0}, h and m do not depend either on temperature T or on time τ.

Let us introduce the symbols

$$hB_{A_0} = Q_{max};$$
$$(m-1)\, B_{A_0}^{m-1} k(T) = A.$$

The physical sense of these complexes is obvious: Q_{max} can be interpreted as the maximum possible heat liberation of a specific concrete; A is the heat liberation rate factor. We note that Q_{max} is independent of temperature, while A = A(T). Then formulas (2-25) and (2-26) can be rewritten as

$$Q = Q_{max}\left\{1 - \left[1 + \int_0^{\tau} A(T)\, d\tau\right]^{-\frac{1}{m-1}}\right\}; \qquad (2\text{-}27)$$

$$q = \frac{Q_{max}}{m-1} A(T)\left[1 + \int_0^{\tau} A(T)\, d\tau\right]^{-\frac{m}{m-1}}. \qquad (2\text{-}28)$$

Suppose τ_1 and τ_2 are moments of equal heat liberation ($Q_1 = Q_2$) of concrete of a predetermined composition, the solidification of which occurs at two different temperature modes T_1 and T_2. Then the ratio of specific intensities of heat liberation at these moments in time q_1 and q_2 are equal to

$$\frac{q_1}{q_2} = \frac{A_1(T_1)}{A_2(T_2)} = \frac{k_1(T_1)}{k_2(T_2)}.$$

Equality of heat liberation $Q_1 = Q_2$ means equality of the integrals

$$\int_0^{\tau_1} k_1(T_1)\, d\tau = \int_0^{\tau_2} k_2(T_2)\, d\tau. \qquad (2\text{-}29)$$

Under isothermal conditions of curing (T_1 = const and T_2 = const), $k_1(T_1) = k_1$ = const; $k_2(T_2) = k_2$ = const.

Therefore

$$k_1\tau_1 = k_2\tau_2,$$

i.e.,

$$\frac{\tau_1}{\tau_2} = \frac{k_2}{k_1} = \frac{q_2}{q_1} = \text{const.} \tag{2-30}$$

From this the following regulary ensues: under isothermal curing conditions of a given concrete, the relationship between the times of equal heat liberation remains constant, determined only by the difference in isothermal curing temperatures.

If we assume that $T_1 = T_a$ is a certain standard temperature, then in terms of the temperature-time function $f_a(T)$, which we mentioned earlier, we have:

$$\tau_{cT} = \frac{k_2}{k_1}\tau = f_a(T)\,\tau.$$

Here by T we refer to an arbitrary but constant temperature.

I. D. Zaporozhets, based on processing of experimental data on the heat liberation of a number of concretes, came to the conclusion that the formula of Rastrup (2-23) is the best analytic expression for the temperature-time function.

Thus,

$$f_a(T) = 2^{\frac{T-T_a}{\varepsilon}}, \tag{2-31}$$

where ε is the characteristic temperature difference leading to doubling of the reaction rates at moments in time such that $T - T_a = \varepsilon$.

If we assume as the standard temperature T_a = 20 C, then

$$\tau_{20} = 2^{\frac{T-20}{\varepsilon}}\,\tau \tag{2-32}$$

and the heat liberation rate coefficient at temperature T is

$$A_T = 2^{\frac{T-20}{\varepsilon}} A_{20}. \tag{2-33}$$

The temperature-time function $f_{20}(T)$ is essentially the relative rate of heat liberation at temperature T if we take the heat liberation rate at T_a = 20 C as unity (in comparing the rates at moments of equal heat liberation).

Here, in the general case by T we can understand the temperature of the concrete at comparable moments in time with any temperature mode of curing

$$f_{20}(T) = \frac{q_T}{q_{20}} \, (Q_T = Q_{20}). \tag{2-34}$$

Here q_{20} is the intensity of heat liberation at moment τ_{20} where T = 20 C = = const (isothermal curing conditions); q_T is the intensity of heat liberation at moment in time τ where $T = T(\tau)$; τ_{20} and τ are the times of equal heat liberation.

With an arbitrary temperature mode

$$\tau_{20} = \int_0^{\tau} f_{20}(T)\, d\tau = \int_0^{\tau} 2^{\frac{T-20}{\varepsilon}}\, d\tau. \tag{2-35}$$

If the conditions of curing are adiabatic, the temperature of the concrete is:

$$T = T_0 + \frac{Q}{c\gamma}, \tag{2-36}$$

where T_0 is the initial temperature of the concrete mixture.

From this

$$\frac{\tau_1}{\tau_2} = 2^{\frac{T_{01}-T_{02}}{\varepsilon}} = \text{const}. \tag{2-37}$$

Here T_{01} and T_{02} are the initial temperature of concrete mixtures 1 and 2; τ_1 and τ_2 are the corresponding times of equal heat liberation.

Thus, under adiabatic curing conditions of a given concrete, the ratio between times of equal heat liberation remains constant, determined by the difference in initial temperatures of the concrete mixture.

The regularity of the times of equal liberation, expressed by formulas (2-32), (2-35) and (2-37), allows us to convert heat liberation from one temperature mode to another. In particular, if we know the heat liberation with any given constant temperature, we can construct a set of heat liberation curves under isothermal curing conditions; similarly, based on data on the heat liberation of concrete under adiabatic conditions with a single given initial temperature of the concrete mixture, we can construct the corresponding set of "adiabatic" curves.

The heat liberation function Q and heat liberation intensity function q, considering the basic results outlined above and the symbols we have introduced, can be represented as

$$Q = Q_{max}\left\{1 - \left[1 + A_{20}\int_0^\tau 2^{\frac{T-20}{\varepsilon}}\,d\tau\right]^{-\frac{1}{m-1}}\right\}; \quad (2\text{-}38)$$

$$q = \frac{Q_{max}}{m-1}A_{20}2^{\frac{T-20}{\varepsilon}}\left[1 + A_{20}\int_0^\tau 2^{\frac{T-20}{\varepsilon}}\,d\tau\right]^{-\frac{m}{m-1}}. \quad (2\text{-}39)$$

From the computational standpoint, we have here 4 parameters: Q_{max}, A_{20}, ε and m.

Based on analysis of data on the heat liberation of concretes, I. D. Zaporozhets accepts: $\varepsilon = 10$, $m = 2.2$.

Then

$$Q = Q_{max}\left\{1 - \left[1 + A_{20}\int_0^\tau 2^{\frac{T-20}{10}}\,d\tau\right]^{-0,833}\right\}; \quad (2\text{-}40)$$

$$q = 0.833Q_{max}A_{20}2^{\frac{T-20}{10}}\left[1 + A_{20}\int_0^\tau 2^{\frac{T-20}{10}}\,d\tau\right]^{-1,833}. \quad (2\text{-}41)$$

The following method has been suggested to determine the remaining two parameters Q_{max} and A_{20}.

Let us assume that we know the heat liberation of a concrete at any given temperature mode (for example in testing of concretes by the thermos method, or in analysis of actual observations of the temperature mode of large concrete masses). Depending on the specific curing conditions, using relationships (2-23), (2-35) or (2-37), we can convert the known experimental data to a standard curve of isothermal heat liberation at T = 20 C = const.

According to I. D. Zaporozhets, this curve is approximated by the expression

$$Q = Q_{max}\{1 - [1 + A_{20}\tau]^{-0.833}\}.$$

Determination of the unknowns Q_{max} and A_{20} can be undertaken using the method of least squares (for example, see the work of L. P. Trapeznikov [127]).

For approximate calculations involved in the prediction of heat liberation of concretes according to I. D. Zaporozhets we can base ourselves on the following considerations.

The quantities h, m, k_{20} and ε are thermochemical characteristics of the cement, and can be considered unchanged for concretes and solutions based on a single cement.

As was noted above, the values of m and ε have more commonality, being $m \simeq \simeq 2.2$ and $\varepsilon \simeq 10$ for all concretes based on Portland cement clinker.

The value of coefficient h, kcal/kg for Portland cement should be calculated on the basis of the mineralogical composition using the formula

$$h = 6.623\ C_3S + 2.839\ C_2S + 8.502\ C_3A + 1.811\ C_4AF,$$

where C_3S, C_2S, C_3A, C_4AF represent the content of the minerals in the clinker, %.

To determine the value of the initial preserve of active water B_{A_0}, kg/m^3, in Portland cement clinker-based concretes, we can use the empirical dependence

$$B_{A_0} = 0.17\ U + 3.8\ OK,$$

where U is the consumption of cement, OK is the cone slump of the concrete mixture.

This formula can be used within the limits of the following values of cement consumption and cone slump:

U, kg/m^3	250	250-200	>200
OK, cm	2-12	2-8	2-4

The value of the coefficient k_{20} for Portland cement is $(2\text{-}6) \cdot 10^{-3}$, while A_{20} is approximately 0.01-0.015 1/hr.

Based on the data presented, it is possible to predict the values

$$Q_{max} = hB_{A_0} \text{ and } A_{20} = 1.2\ B_{A_0} k_{20},$$

allowing us in approximate determinations of heat liberation in concrete to use formulas (2-38) and (2-39).

As we have noted, in the theory of I. D. Zaporozhets outlined above, it is assumed that

1) The total heat liberation in the concrete Q_{max} is independent of the temperature mode of curing;

2) The rate of heat liberation doubles with an increase in temperature by the characteristic temperature difference $\varepsilon \simeq 10$.

However, it has been established in many laboratories and under natural conditions that as the initial temperature of the concrete mixture drops, the total heat liberation of many water engineering concretes increases (see Figure 2-2). Furthermore, concretes are known in which the exothermy is practically independent of temperature (for example, see data on the adiabatic rise in temperature of Japanese concretes based on cements with moderate heat liberation [171]).

The use of the theory of I. D. Zaporozhets or describe the heat liberation of such concretes encounters certain difficulties. In spite of this, the theory of I. D. Zaporozhets is at the present time one of the most complete theories developed, yielding satisfactory models for the description of concrete heat liberation.

Analysis of experimental data has shown that for most concretes, the temperature has a significant influence on the heat liberation when the material is tested under isothermal conditions.

Under adiabatic conditions of curing, after 3 to 6 days the heat liberation equalizes, regardless of the initial temperature of the concrete mixture; this conclusion was reached by V. V. Stol'nikov, R. Ye., Litvinov and A. A. Borisov [118]. P. I. Vasil'yev [15] notes, "Calculations performed at the All-Union Scientific Research Institute for Hydraulic Engineering have shown the significant influence of concrete temperature on the exothermic temperature rise during the first period covering the first 2 to 5 days. Subsequently, this influence becomes insignificant. Therefore, in practical calculations the exothermy can quite frequently be calculated as a fixed function of time."

A considerable volume of the concrete masses in water engineering structures during the stage of construction finds itself under conditions which are near adiabatic. It therefore seems logical to use the suggestion of a number of authors (A. V. Belov, P. I. Vasil'yev, etc.) of performing thermal calculations for the curve of adiabatic temperature rise of concrete of the given composition at an initial temperature equal to the initial temperature of the concrete mixture as it is poured or a certain mean temperature of the process.

In connection with this, experimental data on the heat liberation of concretes of actual compositions at temperatures close to those assumed to be present in the mass, as well as phenomenological methods of construction of heat liberation functions and heat liberation intensity based on these data, become significant.

The influence of temperature on the process of heat liberation is usually studied in testing of concrete specimens cured either under isothermal or under adiabatic conditions[1].

Due to the difficulties arising in the performance of long-term experiments, we are usually limited to 2 or 3 (rarely 4) values of initial temperatures of specimens.

We present below a method of processing experimental data [93] allowing us to calculate the heat liberation curve of concrete with any initial temperature, which is contained at least between the two limiting values of initial temperatures of specimens tested.

Let us present the heat liberation intensity function in the following form for the processing of experimental data on heat liberation of concrete curing under either isothermal or adiabatic conditions:

$$q = q_\nu (1 + b_\nu T) e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, i), \tag{2-42}$$

[1]From here on, the various temperature constants of concretes cured under isothermal conditions and the various initial temperatures of concrete mixtures cured under adiabatic conditions will be called the initial temperatures for brevity.

where q_ν, b_ν, m_ν are empirical coefficients.

In the general case, the constant q_ν, b_ν and m_ν should be limited both as to time interval and as to temperature interval. However, considering the relatively slight differences in initial temperatures of the concrete, we can be satisfied with limiting function (2-42) only as to the time interval.

Thus, we assume that q_ν, b_ν and m_ν are piecewise constant functions of time, defined in $(\tau_{\nu-1}, \tau_\nu)$, i.e., are equai to

$$q_1,\ b_1,\ m_1 \text{ where } 0 < \tau < \tau_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$q_i,\ b_i,\ m_i \text{ where } \tau_{i-1} < \tau < \tau_i.$$

Suppose the curing of the concrete occurs under adiabatic conditions.

With initial temperature T_0 constant through the volume, at any moment in time temperature T is equal to:

$$T = T_0 + \sum_{\nu=1}^{i} \Delta T_\nu, \tag{2-43}$$

where ΔT_ν is the adiabatic rise in temperature during the course of the νth time interval.

But

$$\frac{d(\Delta T_\nu)}{d\tau} = \frac{q_\nu b_\nu}{c\gamma} e^{-m_\nu \tau} \Delta T_\nu + \frac{q_\nu b_\nu}{c\gamma}\left(\frac{1}{b_\nu} + T_{\nu-1}\right) e^{-m_\nu \tau};$$
$$\Delta T_{\nu\,|\,\tau=\tau_{\nu-1}} = 0.$$

From which

$$\Delta T_\nu = \left(\frac{1}{b_\nu} + T_{\nu-1}\right)\left\{\exp\left[\frac{q_\nu b_\nu}{m_\nu c\gamma}\left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu \tau_\nu}\right)\right] - 1\right\},$$

where $T_{\nu-1}$ is the temperature of the concrete at the beginning of the νth time interval.

The temperature of an adiabatically isolated volume of concrete is determined by formula (2-43). As will be shown below, it can also be represented as

$$T = T_0 + e^{\sum_{\nu=1}^{i} g_\nu} \sum_{\nu=1}^{i} \left(\frac{1}{b_\nu} + T_0 \right) e^{-\sum_{\nu'=1}^{\nu} g_\nu} \left[e^{2\rho_\nu \operatorname{sh}(m_\nu \tau_-)} - 1 \right], \tag{2-44}$$

where

$$g_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma} \left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu \tau_\nu} \right);$$

$$\rho_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_+}; \quad \tau_+ = \frac{\tau_\nu + \tau_{\nu-1}}{2}; \quad \tau_- = \frac{\tau_\nu - \tau_{\nu-1}}{2}.$$

Let us assume that we have n curves of adiabatic temperature rise, produced at n different initial temperatures of the concrete mixture.

Let us assign each time sector the index ν ($\nu = 1, 2, \ldots, i$), the curves -- the index s (s = 1, 2,..., n), the experimental points in the sector $(\tau_{\nu-1}, \tau_\nu)$ -- the index k (k = 1, 2,..., m).

Then for each time sector $(\tau_{\nu-1}, \tau_\nu)$ we have a system of equations

$$\Delta T_{\nu k}^{(s)} = \left(\frac{1}{b_\nu} + T_{\nu-1}^{(s)} \right) \left\{ \exp \left[\frac{q_\nu b_\nu}{m_\nu c\gamma} \left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu \tau_k} \right) \right] - 1 \right\}$$

$$(\nu = 1, 2, \ldots, i, \; k = 1, 2, \ldots, m, \; s = 1, 2, \ldots, n), \tag{2-45}$$

from which we are to determine the values of the quantities q_ν, b_ν and m_ν.

The solution of system (2-45) is produced by the method of successive approximations.

Let us assign certain initial values of the parameters $q^{(0)}$, $m^{(0)}$ and $b^{(0)}$. We assume:

$$q = q^{(0)} + \Delta q^{(0)}; \; b = b^{(0)} + \Delta b^{(0)}; \; m = m^{(0)} + \Delta m^{(0)}.$$

Let us substitute these values into equation system (2-45), expand the right portion into a Taylor series and limit ourselves to linear terms. We then have the system

$$\Delta T_k(q^{(0)}, b^{(0)}, m^{(0)}) + \left(\frac{\partial \Delta T_k}{\partial q}\right)_0 \Delta q^{(0)} + \left(\frac{\partial \Delta T_k}{\partial b}\right)_0 \Delta b^{(0)} + \left(\frac{\partial \Delta T_k}{\partial m}\right)_0 \Delta m^{(0)} = \Delta T_k. \tag{2-46}$$

The values of the three corrections $\Delta q^{(0)}$, $\Delta b^{(0)}$, $\Delta m^{(0)}$ from system (2-46) are established by the method of least squares. Suppose these values are $\overline{\Delta q}^{(0)}$, $\overline{\Delta b}^{(0)}$, $\overline{\Delta m}^{(0)}$. We form:

$$q^{(\prime)} = q^{(0)} + \overline{\Delta q}^{(0)}, \quad b^{(\prime)} = b^{(0)} + \overline{\Delta b}^{(0)}, \quad m^{(\prime)} = m^{(0)} + \overline{\Delta m}^{0}$$

and assume

$$q^{(\prime\prime)} = q^{(\prime)} + \Delta q^{(\prime)}, \quad b^{(\prime\prime)} = b^{(\prime)} + \Delta b^{(\prime)}, \quad m^{(\prime\prime)} = m^{(\prime)} + \Delta m^{(\prime)}.$$

Determination of $\Delta q^{(\prime)}$, $\Delta b^{(\prime)}$, $\Delta m^{(\prime)}$, as before, is performed by the method of least squares.

The process of approximation is interrupted when the subsequent values of $q_\nu^{(j)}$, $b_\nu^{(j)}$, $m_\nu^{(j)}$ differ little enough from the preceding values $q_\nu^{(j-1)}$, $b_\nu^{(j-1)}$, $m_\nu^{(j-1)}$.

Similarly, we process the experimental data produced for concrete curing under isothermal conditions.

The system of equations from which with fixed ν we determine q_ν, b_ν and m_ν in this case becomes:

$$\Delta Q_{\nu k}^{(s)} = \frac{q_\nu b_\nu}{m_\nu}\left(\frac{1}{b_\nu} + T_s^{(s)}\right)\left[\exp(-m_\nu \tau_{\nu-1}) - \exp(-m_\nu \tau_k)\right]$$

$$(\nu = 1, 2, \ldots, i, \; k = 1, 2, \ldots, m, \; s = 1, 2, \ldots, n), \tag{2-47}$$

where $\Delta Q_{\nu k}^{(s)}$ is the heat liberation in the νth time interval on curve s from the beginning of sector $\tau_{\nu-1}$ to moment τ_k; $T_b^{(s)}$ is the temperature of the concrete.

TABLE 2-10. TEMPERATURE OF CONCRETE CURED UNDER ADIABATIC CONDITIONS WITH VARIOUS INITIAL TEMPERATURES OF THE SPECIMEN (PURE CLINKER PORTLAND CEMENT GRADE 400, CEMENT CONSUMPTION 250 kg/m^3)

1 Сроки тверде-ния, сутки	2 № временного участка	3 Начальная температура образца, °С			
		7		21,5	
		4 Опытные данные [114]	5 Расчетные значения	4 Опытные данные [114]	5 Расчетные значения
0,5	1	—	18,7	39,3	—
1	2	28,2	27,8	47,0	46,8
2	3	37,3	36,8	54,2	54,0
3	4	41,8	41,3	57,4	57,2
4	5	44,5	44,2	60,1	59,9
5	6	46,3	45,9	61,9	61,8
7	7	47,2	46,8	62,6	62,8
14	8	52,2	51,9	67,4	67,3

Key: 1, Curing Time, Days; 2, Number of Time Sector; 3, Initial Temperature Specimen, C; 4, Experimental Data of [114]; 5, Calculated Values

Based on the method just described, a program was written for computer processing of experimental data on the heat liberation of concrete cured under either adiabatic or isothermal conditions. This program was used for analysis of data on the heat liberation of a number of domestic and foreign water engineering concretes and cements.

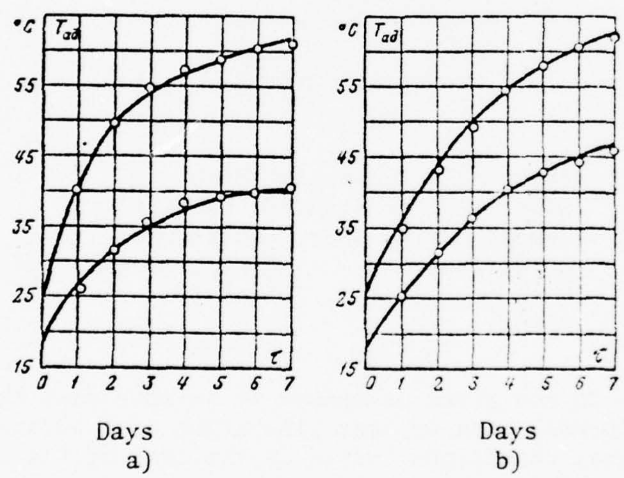

Figure 2-3. Temperature of Concrete Cured Under Adiabatic Conditions with Various Initial Temperatures and Compositions of the Concrete Mixture. a, Grade 400 Portland Cement, Cement Consumption 250 kg/m^3, W/C = = 0.58; b, Puzzolan Portland Cement with Tripolite Grade 400, Cement Consumption 300 kg/m^3, W/C = 0.60; -----, Experimental Curves [29]; 0, Calculated Points

Some results of calculation are presented in Table 2-10 and Figures 2-3 and 2-4.

The 0 approximations in solution of equation system (2-45) or (2-47), practical convergence of the sequence of approximations, as well as the duration of time sectors were established by the method of testing. Usually with a little skill, one or two versions are sufficient to achieve the goal. In our studies, various values of approximation led to the same values of parameters q_ν, b_ν and m_ν.

The method suggested is suitable for processing of experimental data produced with various (no less than 2) initial temperatures of the specimens. In order to estimate the reliability of the calculation data on heat liberation with initial temperatures of the concrete mixture lying with the interval of initial temperatures studied, the following numerical experiment was performed. It was based on the results of the experiments of S. Takano [171] on the heat liberation of concrete with normal Portland cement. Processing was performed according to 3 versions (version I, the basic version -- initial temperatures of 5, 10, 20, 30 C; version II -- initial temperatures 5, 30 C; version III -- initial temperatures 10, 20 C). The calculations showed that the data for all 4 curves produced in versions II and III differ from the corresponding data of version I by no more than 1-2 C.

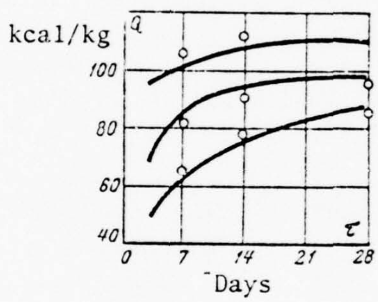

Figure 2-4. Specific Heat Liberation of Standard Portland Cement Curing Under Isothermal Conditions. - - - -, Experimental Data of [155]; 0, Calculated Points

What we have said up to now gives us reason to believe that the method of processing of experimental data on heat liberation of concretes curing under adiabatic or isothermal conditions, based on the idea of the heat liberation intensity function

$$q = q_\nu (1 + b_\nu T) e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, i).$$

where q , b and m are empirical coefficients, piecewise-constant functions of time, yields satisfactory results.

Let us write the expression for the intensity function of heat liberation in a somewhat more general form

$$q = q_\nu (d_\nu + b_\nu T) e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, i), \tag{2-48}$$

where q_ν, d_ν, b_ν and m_ν are parameters, piecewise-constant functions of time, defined in $(\tau_{\nu-1}, \tau_\nu)$.

It is not difficult to see that formula (2-48), in a certain sense, summarizes the suggestions of various authors.

Let us analyze some of them.

1. The intensity of heat liberation is constant $q = q_0$ = const. We assume:

$$\nu = 1, \quad d_\nu = 1, \; b_\nu = 0, \quad m_\nu = 0.$$

2. The intensity of heat liberation is a piecewise-constant function of time $q = q_\nu$. We assume:

$$\nu = 1, 2, \ldots, i; \quad d_\nu = 1, \quad b_\nu = 0, \quad m_\nu = 0.$$

A function of this form can be used to approximate any tabular or graphic description of the "heat liberation-time" curve.

3. The intensity of heat liberation depends only on time $q = q(\tau)$. Here, two versions are possible:

a) $q = q_0 e^{-m\tau}$.

We assume:

$$\nu = 1, \quad d_\nu' = 1, \quad b_\nu = 0, \quad m_\nu = m.$$

The sum of exponents such as

$$q = \sum_{\nu=1}^{i} q_\nu e^{-\nu m \tau}$$

does not change the essence of the concept.

b) $q = q_\nu e^{-m_\nu \tau}$.

We assume:

$$\nu = 1, 2, \ldots, i; \quad d_\nu = 1, \quad b_\nu = 0.$$

As was noted above, these concepts allow us to approximate any "heat liberation-time" curve.

4. The intensity of heat liberation is described by a function of S. V. Aleksandrovskiy (2-18)

$$q = \omega(\tau) T.$$

We assume:

$$\nu = 1, 2, \ldots, i; \quad d_\nu = 0, \quad b_\nu = 1,$$

i.e., approximate the function $\omega(\tau)$ by a form such as $q_\nu e^{-m_\nu \tau}$.

5. The realization of the suggestion of indirect consideration of the dependence of heat liberation on temperature by the use in our calculations of the curve of "adiabatic temperature rise versus time" at an initial temperature equal to the temperature at which the concrete is poured into the actual object or a certain mean temperature of the process, is reduced to construction of this curve according to the method suggested and approximation of the curve either with the form

$$q = q_\nu e^{-m_\nu \tau}$$

or

$$q = \sum_{\nu=1}^{i} q_\nu e^{-\nu m \tau}.$$

6. With time (τ great), heat liberation in the concrete stops. The exothermy is not observed also in those areas of the concrete body where the temperature is below a certain limiting negative temperature. In these cases, it is sufficient to assume $q_\nu = 0$.

In addition to the above, we can recommend function (2-48), the parameters of which q_ν, d_ν, b_ν and m_ν were defined from the "adiabatic" experiments, for calculation of the temperature field of concrete water engineering structures, which are erected with the observation of the necessary technical conditions (sufficient heat insulation surface of the concrete at negative temperatures of the medium, etc.).

Indirect proof of the correctness of these recommendations can be found in the results presented below from comparison of temperature fields of concrete masses calculated on the assumption that:

1) The heat liberation intensity function is determined by a formula such as

$$q = q_\nu (1 + b_\nu T) e^{-m_\nu \tau},$$

where the parameters q_ν, b_ν, m_ν and the duration of time sectors ($\tau_{\nu-1}$ τ_ν) are established from experimental data based on the heat liberation of concrete curing under adiabatic conditions at various initial specimen temperatures;

2) The heat liberation intensity function depends only on time according to the expression

$$q = q_\nu e^{-m_\nu \tau},$$

The parameters q_ν, m_ν and length of time sectors ($\tau_{\nu-1}$, τ_ν) are established from the curve of adiabatic heat liberation at an initial temperature equal to the temperature at which the concrete is poured.

It is not difficult to see that part 2 realizes the suggestion of A. V. Belov, P. I. Vasil'yev, etc.

The temperature field of a concrete wall with a width R = 12 m with asymmetrical boundary conditions of the third kind was studied. It was assumed that exothermy of the concrete corresponds to the curves of adiabatic temperature rise presented in [114].

Calculations were performed with various values of heat transfer coefficient α (0.65-4.0 kcal/(m^2·hr·C)) and ambient temperatures T_1 and T_2 (+17- -20 C).

As we can see from Table 2-11, in the characteristic cases studied, consideration of heat liberation by our method and based on the curves of adiabatic temperature rise at the initial temperature, equal to the initial temperature of the concrete mixture, yields similar results.

TABLE 2-11. TEMPERATURE OF SURFACE OF CONCRETE WALL, C, WITH VARIOUS METHODS OF CALCULATION (WITH THE WALL R = 12 m, INITIAL TEMPERATURE T_0 = 10 C)

1 Сроки твердения, ч	2 Учет тепловыделения по методике автора		3 Учет тепловыделения на основе начальной температуры бетона	
	4 Координаты			
	$x=0$	$x=R$	$x=0$	$x=R$
$T_{1\,(x=0)} = T_{2\,(x=R)} = -20$ C; $\alpha_{1\,(x=0)} = 1{,}5$ [5] ккал/(м²·ч·°С); $\alpha_{2\,(x=R)} = 1{,}0$ [5] ккал/(м²·ч·°С)				
24	21,5	24,5	21,3	24,4
72	25,6	30,8	24,6	30,0
168	20,5	27,5	20,3	27,5
216	18,7	26,2	18,5	26,0
$T_{1\,(x=0)} = T_{2\,(x=R)} = -20$ C; $\alpha_{1\,(x=0)} = 2{,}0$ [5] ккал/(м²·ч·°С); $\alpha_{2\,(x=R)} = 4{,}0$ [5] ккал/(м²·ч·°С)				
24	18,9	10,7	18,6	10,3
72	21,4	9,6	20,2	8,2
168	15,0	2,2	14,9	2,0
336	10,7	−1,6	10,3	−2,0
$T_{1\,(x=0)} = 17$ C, $T_{2\,(x=R)} = -20$ C; $\alpha_{1\,(x=0)} = 4{,}0$ [5] ккал/(м²·ч·°С); $\alpha_{2\,(x=R)} = 0{,}65$ [5] ккал/(м²·ч·°С)				
24	27,2	26,9	26,8	26,8
72	30,8	35,0	30,3	34,4
168	28,0	33,8	28,1	33,8
336	26,9	32,3	26,8	32,3

Key: 1, Curing Times, hr; 2, Consideration of Heat Liberation by Method of Author; 3, Consideration of Heat Liberation by Initial Temperature of Concrete; 4, Coordinates; 5, kcal/(m^2.hr·C)

Note. $T_{1(x=0)}$ and $T_{2(x=R)}$ is the temperature of the wall surface; $\alpha_{1(x=0)}$ and $\alpha_{2(x=R)}$ are the heat exchange coefficients.

2-3. Heat Exchange of Concrete Surfaces with the Environment

In stating the problem of heat conductivity, heat exchange of the body with the environment is generally considered using boundary conditions of the third kind

$$\left.\frac{\partial T}{\partial n}\right|_{\Gamma} = h\left[\varphi(\mathcal{P}, \tau) - T|_{\Gamma}\right]. \tag{2-49}$$

Here n is an external perpendicular to the boundary surface Γ at point $\mathcal{P}$; $\varphi(\mathcal{P}, \tau)$ is the ambient temperature; $h = \alpha/\lambda$ is the relative heat transfer factor; α is the heat transfer coefficient.

The surface of concrete water engineering structures and their elements contact the external air and water.

This contact is either directly or through a deck (metal, wood, insulating, etc.), or through a layer of thermal insulation, etc.

In addition to these factors, solar radiation may also influence the heat exchange of concrete masses.

Ambient Air Temperature

In predicting the temperature mode of water engineering structures, the initial climatological data generally used are the materials of observation of weather stations located near the region of construction.

In Table 2-12 we present the mean monthly and mean annual temperatures in the regions of construction of the most important hydrosystems of the USSR.

Analyses of a number of investigators [18, 172] have shown that the air temperature is satisfactorily described by the sum of three simple harmonic functions:

annual with a period of 365 days (8760 hr) constructed from the mean monthly temperatures

$$T_a = T_{ma} + A_a \sin (\omega_a \tau + \varepsilon_a),$$

diurnal with a period of 1 day (24 hours), constructed from the mean hourly temperatures

$$T_d = T_{md} + A_d \sin (\omega_d \tau + \varepsilon_d),$$

semimonthly with a period of 15 days (360 hr), constructed from the mean daily temperatures

$$T_m = T_{mm} + A_m \sin (\omega_m \tau + \varepsilon_m).$$

Here T_m is the mean temperature (mean annual, etc.); A is the amplitude; $\omega = 2\pi/\theta$ is the cyclical frequency; θ is the period; ε is the initial phase.

Sometimes, we also introduce the quarterly harmonic with a period of 2190 hr, constructed from the mean monthly temperatures.

TABLE 2-12. AIR TEMPERATURE, C, IN THE REGIONS OF CONSTRUCTION OF CERTAIN HYDROSYSTEMS

Hydrosystem	Month												Mean Annual
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	
Bukhtarmin-skiy	—18,3	—15,2	—6,1	4,0	13,3	17,9	20,0	17,7	11,3	3,7	—6,1	—14,5	0
Bratskiy	—23,8	—21,1	—11,6	—1,0	7,3	15,0	18,2	15,2	7,8	—1,4	—13,0	—22,2	—2,6
Krasnoyarskiy	—20,2	—18,4	—9,5	0,5	9,0	16,0	18,8	15,2	9,1	2,1	—10,2	—17,4	—0,4
Zeyskiy	—31,6	—25,1	—15,7	—2,0	9,1	14,4	18,6	16,3	9,6	—2,3	—18,4	—26,7	—4,5
Ust'Ilimskiy	—25,6	—22,9	—13,0	—2,6	6,2	14,2	17,8	14,7	7,1	—2,4	—15,1	—24,8	—3,9
Kolskiy	—37,9	—35,1	—26,4	—13,4	2,0	12,6	15,5	12,4	4,1	—12,1	—28,0	—37,1	—12,0
Toktogul'skiy	—11,1	—6,3	5,1	14,9	20,4	23,2	26,3	26,3	21,9	13,9	6,4	—2,0	11,6
Ingurskiy	5,2	5,5	7,3	12,6	17,2	20,2	22,2	22,7	19,6	15,6	11,7	7,6	13,9

The number of harmonics necessary to describe the air temperature is obviously determined by the climatic conditions in the region of construction and is established individually for each specific hydrosystem.

Depending on the conditions of construction and operation of structures and their elements, as well as the purposes of the investigation, we may also assign other rules of change of air temperature. Thus, during the construction period, for example during the performance of concrete pouring operations in the winter time beneath a tent, the air temperature can be assumed basically constant; the air temperature over the surface of the mass can also be assumed constant during relatively brief time intervals between coverage of a block by a block as a mass is being erected. Satisfactory results are produced by approximating the true course of temperature of the air by a stepped function. For more precise investigations of the thermal mode of concrete masses during the period of construction, it is sometimes necessary to assign the air temperature in tabular form. During the period of use, it is frequently sufficient to represent the air temperature as a function with an annual harmonic.

Water Temperature

The question of proper assignment of water temperature at the boundary with a concrete mass of a water engineering structure is an independent and very complex task from the theory of heat exchange.

The thermal mode of reservoirs, determined by the temperature of the water in the pool above the dam, has been the subject of many investigations by both domestic and foreign specialists.

In the prediction of the thermal mode of reservoirs, the method of analogues has been widely used, based on the data of natural observation of the thermal mode of nearby lakes and reservoirs of similar depth with similar influx and drainage conditions.

Quite satisfactory results are produced by the method of thermal calculation of reservoirs developed by A. I. Pekhovich and his students [88, 89, 90, 128].

The thermal mode of reservoirs in hydrosystems differs in quite a number of points, a result of the peculiarities of the climate of the region of the reservoir, differences in the plan dimensions and depths, degree of water flow-through, etc. In addition to this, we can note several common points in the thermal mode inherent in most reservoirs.

Five periods are usually distinguished in the annual thermal cycle of reservoirs [90, 128]:

Period I -- the spring thaw (to 4 C);
Period II -- the summer warming (from 4 C up);
Period III -- fall cooling (down to 4 C);

Period IV -- the pre-freezing cooling (below 4 C);
Period V -- the winter period (beneath the ice cover).

Three types of reservoirs are distinguished as to depth: shallow, deep and very deep.

Shallow reservoirs are characterized by practically identical temperature throughout their depth. If there is no ice cover, the change in water temperature with time at all depths follows the change in air temperature with a certain time lag, while during the time the reservoir is covered with ice the water temperature generally does not exceed 0.5-1 C.

In deep reservoirs, there is a temperature drop with depth. During the period of heating, the water mass can be divided into three layers: the upper layer, with relatively high temperature, the lower layer with low temperature and an intermediate temperature "jump" layer. The change in temperature with time occurs in all layers, including the lower layer. During the period of ice cover, the temperature drop with depth is 2-3 C.

One distinguishing feature of very deep reservoirs is the slight variability of water temperature in the benthic layer. In the higher layer, the temperature mode is similar to the mode of deep reservoirs.

The type of reservoir as to depth depends not only on the morphometric depth H, but also on certain other factors, among which are turbulent and free convective mixing, characterized by the heat conductivity factor λ. The method of determination of the type of reservoir as to depth is given in the "Instructions for Thermal Calculation of Reservoirs" [128].

We can approximately consider that a reservoir is shallow if the water depth

$$H < 0{,}2\,\frac{\lambda}{\alpha}.$$

Where

$$H > 0{,}2\,\frac{\lambda}{\alpha}$$

the reservoir is either deep or very deep.

The additional condition

$$H^2 > 5 - 10\,\frac{\lambda\tau}{c\gamma}$$

distinguishes a very deep reservoir.

The water temperature in the reservoir defines the temperature of the upper face of the dam. As concerns those sections of the lower face in contact with the water, we should keep in mind the water temperature downstream.

In the area of dam spillways, the temperature of this sector of the lower face can be considered equal to the water temperature in the pool at the level of the water intake.

In the sections of the dam by the power station, due to the thermal shielding of the dam by the power station buildings, temperature should be considered approximately 5 C higher.

The water temperature in the downstream section along the course of the water flow changes with distance from the downstream edge of the dam and can be calculated by the following formula [128]

$$T = (T_0 - \Phi_{cp})\, e^{-\frac{\alpha S}{Mc\gamma}} + \Phi_{cp},$$

where T and T_0 are the water temperature at the design and initial reservoir lines respectively; Φ_m is the effective air temperature (considering possible absorption of solar radiation); α is the heat transfer factor of the water surface; S is the surface area between the initial and calculated dam lines; $c\gamma$ is the specific volumetric heat capacity of the water; M is the water consumption.

In order to illustrate this point, let us briefly analyze the thermal mode of the Bratsk reservoir.

The reservoir of the Bratsk Hydroelectric Power Plant is a deep-type reservoir with little water flow-through. The full volume of the reservoir is about 179 billion m^3, the surface area is 5,500 km^2, the mean depth is 32.6 m. In the area of the reservoir near the dam, the depth of the water in which reaches 100 m, constant thermal observations have been performed since 1961 by the Hydroelectric Power Planning Institute imeni S. Ya. Zhuk. Analysis of the results of these observations can be found in the special literature.

The vertical distribution of water temperature in the Bratsk Reservoir adjacent to the dam in 1964 is illustrated by Figure 2-5.

Opening of the Bratsk Reservoir (beginning of period I of the annual thermal cycle) usually occurs in early May. The water temperature rises, while the vertical temperature gradients are practically equal to 0. After 5 or 10 days following complete thawing of the ice (mid-June), the water temperature reaches 4 C.

In mid-June, the water mass of the reservoir is rapidly stratified vertically and period II of the annual thermal cycle begins. The change in water temperature with time is characterized by a rise in temperature, the rise rate of temperature with time decreasing with depth. In the beginning of the period of heating, the temperature jump layer is located in the upper 5 meters, at the end of the period of warming -- at a depth of 25-30 m. It should be noted that at this time, in addition to the general rise in heat content of the water mass, brief drops in temperature are observed, extending down to depths of 30-40 m. By mid-August, the water temperature reaches its maximum.

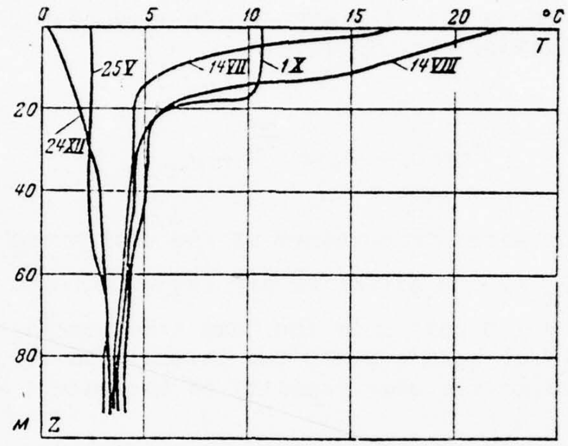

Figure 2-5. Vertical Distribution of Water Temperature in the Section of the Bratsk Reservoir Near the Dam in 1964

Beginning in the second half of August, the water mass of the Bratsk reservoir begins to cool (period III of the annual thermal cycle). In the upper layer of the water, the temperature gradients gradually decrease, then the water temperature down to a depth of 20-30 m is practically equalized and the rate of temperature fall increases. In mid-October, free convection is reinforced and rapidly reaches through the entire thickness of the water to the bottom. The rate of temperature fall at all depths is identical.

Free convection ends in the last third of November (period IV of annual temperature cycle). As a result of this, the delivery of heat from the deep layers of water to the surface is greatly reduced and the first ice forms. The temperature in the upper layers of water continues to drop rapidly and, in late November-early December, the ice cover is formed, i.e., period V of the annual temperature cycle starts. From this moment, the temperature of the water at all depths changes very little.

Heat Transfer Coefficient

The conclusion of the boundary condition of the third kind (2-49) is based on the assumption that the heat flux density at the surface of the body W_n is proportional to the temperature difference between the environment T_c and the surface of the body T_n

$$W_n = \alpha (T_c - T_n). \tag{2-50}$$

This position is known as the rule of convective heat exchange of Newton.

It follows from Newton's rule that the heat transfer coefficient α is numerically equal to the heat flux density at the "body-environment" division surface, related to the temperature difference between the surface and the environment.

The unit of measurement of the heat transfer coefficient is: engineering system -- kcal/($m^2 \cdot hr \cdot C$), SI -- w/($m^2 \cdot C$), conversion factor -- 1 kcal/($m^2 \cdot hr \cdot C$) = 1.1630 w/($m^2 \cdot C$).

A distinction is made between the local heat transfer coefficient, characterizing the heat exchange at a given point on the surface, and the mean heat transfer coefficient for the surface S.

In this book, we will utilize the concept of the mean heat transfer coefficient (though we will omit the term "mean"). This allows us to assume the heat transfer coefficient constant over the surface area being averaged.

Usually the heat transfer coefficient α is assumed equal to

$$\alpha = \alpha_c + \alpha_{rad}, \tag{2-51}$$

where α_c is the heat transfer coefficient due to convection; α_{rad} is the heat transfer coefficient due to radiation.

We will analyze the convection heat transfer coefficient α_c at this point, leaving the radiation heat transfer coefficient α_{rad} for later.

The value of α_c depends on the form and dimensions of the surface of the body, the nature and speed of movement of the medium above the surface, temperature and other factors.

In textbooks and handbooks on heat transfer [62, 72, 79], the following formula is recommended for calculation of α_c of open flat (or near flat)

surfaces with forced turbulent movement of air:

$$Nu = 0.032 \, Re^{0.8}, \tag{2-52}$$

where $Nu = \alpha_c \ell/\lambda$ is the Nusselt criterion; $Re = v\ell/\nu$ is the Reynold's number; α_c is the convection heat transfer coefficient, kcal/(m^2·hr·C); λ is the heat conductivity coefficient of the air (considering the air temperature), kcal/(m·hr·C); ν is the kinematic viscosity of the air, m^2/s; ℓ is the defining dimension, m; the defining dimension used is the length of a plate in the direction of movement of the air.

Formula (2-52) was produced by M. A. Mikheyev [79] as a result of processing of the experimental data of Jurges and Frank by methods of the theory similarity.

The experiments of Jurges and Frank were performed on plates measuring 0.5 x 0.5 m and 0.7 x 0.7 m, and therefore the correctness of application of formula (2-52) for calculation of α_c in the case of surfaces of significantly greater dimensions (up to 50-100 m) as are encountered in water engineering constructions is doubtful. Equally doubtful is the possibility of using the formulas suggested by Jurges

$$\alpha_c = 4.8 + 3.4 \, v, \ \text{kcal/(m}^2\cdot\text{hr}\cdot\text{C)}; \ v < 5 \text{ m/s},$$

$$\alpha_c = 6.12 \, v^{0.78}, \ \text{kcal/(m}^2\cdot\text{hr}\cdot\text{C)}; \ v > 5 \text{ m/s},$$

which are used by several authors in their calculations.

Based on analysis of the data of calculation, experimental and natural observation, the corresponding norms documents recommend for calculation of the temperature field of concrete masses the standard values of convection heat transfer coefficient α_c of 20 kcal/(m^2·hr.C) for open surfaces and 5-10 kcal/(m^2·hr·C) for surfaces forming a cavity. These values of α_c can be considered preferable at the present time.

The convection heat transfer coefficient α_c on the surface of contact between a solid and water is about 100-200 kcal/(m^2·hr·C). These high values of α allow us to assume in formula (2-49) $\alpha \to \infty$, which is equivalent to assigning a boundary condition of the first kind at the surface of the solid body (temperature of the surface of the body equal to the temperature of the water).

In calculating the temperature fields of concrete masses, the thermal protective properties of the deck (metal, wood heating), heat insulation and other means of thermal protection can be considered by introducing heat transfer coefficient β, defined by the ratio

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{R_d}{\lambda_d},$$

where R_d is the thickness of the deck (insulation); λ_d is the heat transfer coefficient of the deck material; α is the heat transfer coefficient at the surface of the deck.

When the deck (insulation) is heterogeneous and consists of several layers of material, the heat transfer coefficient β is calculated by the formula

$$\frac{1}{\beta} = \frac{1}{\alpha} + \sum_{i=1}^{k} \frac{R_{d\,i}}{\lambda_i}, \tag{2-53}$$

where k is the number of layers in the deck.

When a massive block deck is used (which has been widely used, for example, by Bratskgestroy), this simplification may lead to significant errors in determination of the temperature; therefore, a more precise model of a multilayer, particularly a two-layer, body must be used.

The influence of solar radiation on the temperature mode of concrete water engineering structures has been studied at a number of dams in the USA [172], in the dam of the Bratsk Hydroelectric Power Plant [7], and in construction of the dam of the Toktogul'skaya Hydroelectric Power Plant [119].

Solar radiation (insolation) is the flux of radiant energy reaching the surface of the Earth and consisting of both direct and scattered radiation[1].

Direct radiation is the predominant component of solar radiation, coming directly from the disc of the sun.

Scattered radiation is that component of solar radiation arriving from all points in the sky after scattering in the atmosphere.

The intensity of the total radiation B is:

[1]The composition of solar radiation also includes the natural radiation of the atmosphere. Its influence on the thermal mode of structures is negligible, and therefore this component will not be discussed here.

$$B = J \sin h + D \approx J \sin (1+k),$$

where J and D are the intensities of direct and scattered radiation respectively; h is the angular height of the sun; k is the proportionality factor between scattered and direct radiation at noon.

We know from spherical geometry that:

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t,$$

where ϕ is the geographic latitude; δ is the declination of the sun; t is the hour angle, corresponding to true solar time.

For any specific day

$$\sin h = c + d \cos t,$$

where

$$c = \sin \varphi \sin \delta; \; d = \cos \varphi \cos \delta,$$

and the summary radiation at any moment in time τ in the day

$$B_\tau = J(1+k)(c + d \cos t),$$

while the summary radiation throughout a 24-hour period (or the daily radiation norm [119])

$$B_\Sigma = \int_{\tau'_n}^{\tau''_n} J\left(c + d \cos \frac{\pi}{12}\tau\right) d\tau + \int_{\tau'_p}^{\tau''_p} Jk\left(c + d \cos \frac{\pi}{12}\tau\right) d\tau,$$

where τ'_n and τ''_n are the times of sunrise and sunset; τ'_p and τ''_p are the initial and final times of influence of solar radiation.

The values of J, D and k are determined from actinometric observations by the weather service, h, ϕ and δ from reference tables, τ'_p, τ''_n, τ'_p and τ''_p are established on site.

The value of B_{Σ} for open spaces can be taken from the handbooks.

In order to gain an idea of the order of magnitude of J, D and B, let us present some of the latest results of experimental studies performed on the construction area of the Toktogul'skaya Hydroelectric Power Plant [119]. In mid-June, the intensity of direct solar radiation J = 720 kcal/(m^2·hr), of scattered radiation B = 108 kcal/(m^2·hr).

Where h_{max} = 75.5°, the maximum intensity of summary radiation

$$B_{max} = 720 \cdot 0.968 + 108 = 805 \text{ kcal/(m}^2 \cdot \text{hr)}.$$

The daily norm of solar radiation during the hottest months -- from May through September -- is 7000-7200 kcal/(m^2·day).

Solar radiation absorbed by a concrete surface is:

$$F = \varepsilon_b B - E_b,$$

where ε_b is the degree of blackness of the concrete surface; E_b is the natural radiation of the concrete surface.

With a source temperature (solar disc) of 6200 K, the degree of blackness of the concrete surface ε_b = 0.60.

As a result of studies at the Toktogul'skaya Hydroelectric Power Plant [119], the following values of degree of blackness of a dry concrete surface have been produced: at noon ε_b = 0.55; the mean over the entire time of insolation ε_b = 0.44.

The natural radiation E_b, kcal/(m^2·hr) of a concrete surface into the atmosphere is

$$E_b = 4.96\varepsilon_{eff}\left[A_B\left(\frac{T_\Pi}{100}\right)^4 - \varepsilon_B\left(\frac{T_B}{100}\right)^4\right],$$

where ε_{eff} is the effective degree of blackness of the surface; ε_B is the degree of blackness of the gas (air); A_B is the absorptive capacity of the gas (air); T_B is the air temperature, K; T_Π is the temperature of the concrete surface, K.

For mean conditions of the atmosphere we can assume:

$$\varepsilon_{n} = 0{,}95,\ A_{\text{в}} = 0{,}95;\ \varepsilon_{\text{обб}} = \frac{\varepsilon_{б}+1}{2}.$$

Consequently,

$$E_{б} = 2{,}36\,(\varepsilon_{б}+1)\left[\left(\frac{T_{п}}{100}\right)^{4} - \left(\frac{T_{в}}{100}\right)^{4}\right].$$

In order to formulate the boundary conditions in the case of insolation, we can compose the equation for thermal balance on the surface of the body

$$-\lambda \frac{\partial T}{\partial n}\bigg|_{\Gamma} - \alpha\,(T|_{\Gamma} - T_{в}) + \varepsilon_{б} B - E_{б} = 0,$$

from which it follows that

$$\frac{\partial T}{\partial n}\bigg|_{\Gamma} = h\,[T_{в} - T|_{\Gamma}] + \frac{1}{\lambda}\varepsilon_{б} B - \frac{1}{\lambda}\,2{,}36\,(\varepsilon_{б}+1)\,[\theta|_{\Gamma}^{4} - \theta_{в}^{4}]$$

or

$$\frac{\partial T}{\partial n}\bigg|_{\Gamma} = h\,[\Phi - T|_{\Gamma}] - \frac{1}{\lambda}\,2{,}36\,(\varepsilon_{б}+1)\,[\theta|_{\Gamma}^{4} - \theta_{в}^{4}], \qquad (2\text{-}54)$$

where

$$h = \frac{\alpha}{\lambda};$$

$$\Phi = T_{в} + \frac{\varepsilon_{б}}{\alpha} B;$$

$$\theta|_{\Gamma} = \theta_{п} = \frac{T|_{\Gamma}\,(K)}{100},\quad \theta_{в} = \frac{T_{в}\,(K)}{100}. \qquad (2\text{-}55)$$

Boundary condition (2-54) is nonlinear. For relative low temperatures of the concrete surfaces, such as we encounter, it can be linearized.

The difference in the 4th powers $(\theta_{\pi}^{4} - \theta_{B}^{4})$ is broken down into the factors

$$\theta_{п}^{4} - \theta_{в}^{4} = (\theta_{п}^{3} + \theta_{п}^{2}\theta_{в} + \theta_{п}\theta_{в}^{2} + \theta_{в}^{3})\,(\theta_{п} - \theta_{в}).$$

For the air and surface temperature interval 20-50 C, this relationship can be replaced by the following simpler relationship with an error of not over ±5%.

$$\theta_{\text{н}}^4 - \theta_{\text{в}}^4 \approx 1{,}17\,(T|_\Gamma - T_{\text{в}}).$$

Then the linearized boundary condition, considering solar radiation, is written as

$$\frac{\partial T}{\partial n}\Big|_\Gamma = h_{\text{eff}}\,[\Phi - T|_\Gamma], \tag{2-56}$$

where

$$h_{\text{eff}} = \frac{\alpha}{\lambda}; \tag{2-57}$$

$\alpha = \alpha_c + \alpha_{rad}$; α_c is the convection heat transfer coefficient; α_{rad} is the radiation heat transfer coefficient

$$\alpha_{rad} = 2.76(\varepsilon_b + 1). \tag{2-58}$$

The equivalent temperature of the medium Φ is determined by formula (2-55).

Formulas (2-57) and (2-58) are used to determine α_{rad} in the case when there is no insolation, and the natural radiation of the body cannot be ignored.

2-4. Statement of the Problem of Heat Conductivity for Concrete Masses

In formulating the problem of heat conductivity for concrete masses in water engineering structures, we will base ourselves on the primary results of the preceding sections of this chapter.

1. Concrete can be looked upon as a quasihomogeneous, isotropic material.

2. The heat-physical characteristics of concrete and the rock base -- heat conductivity coefficient λ and temperature conductivity coefficient a -- are independent of temperature and time.

3. In concrete, in the process of its curing, heat is liberated, caused by the exothermic reactions of hydration of cement. The intensity of heat liberation q is a function of time, and in the initial period of curing heat liberation is also influenced by temperature.

For the mathematical description of the dependence of the intensity of heat liberation in concrete on time alone, we can recommend the expressions:

a) Exponential function

$$q = q_0 e^{-m\tau}; \tag{2-59}$$

b) Piecewise-exponential function

$$q = q_\nu e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, i), \tag{2-60}$$

where q_ν, m_ν are parameters, piecewise-constant functions of time defined over $(\tau_{\nu-1}, \tau_\nu)$; i is the number of sectors of subdivision of the "heat liberation-time" curve;

c) The sum of the exponential functions

$$q = \sum_{\nu=1}^{k} q_\nu e^{-\nu m\tau}. \tag{2-61}$$

Exponential function (2-59) in many cases quite closely approximates the curve of heat liberation. Using functions such as (2-60) or (2-61), we can with good accuracy approximate the "heat liberation-time" curve of concrete of any composition.

Consideration of the dependence of heat liberation intensity in concrete on temperature and time can be achieved by one of the following methods:

a) For each zone of a mass with assigned initial temperature, the heat liberation intensity function is constructed as a function only of time on the basis of the curve of adiabatic temperature rise of the concrete in question with the initial temperature equal to the temperature of the concrete mass as it is poured;

b) A generalized function of the intensity of heat liberation is used such as

$$q = q_\nu (d_\nu + b_\nu T) e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, i), \tag{2-62}$$

where q_ν, d_ν, b_ν, m_ν are parameters, piecewise-constant functions of time, defined over $(\tau_{\nu-1}, \tau_\nu)$; the parameters q_ν, d_ν, b_ν, m_ν and the number of

subdivision sectors i are established from the curves of "adiabatic temperature rise versus time" with various (at least two) initial concrete temperatures.

The generalized function of heat liberation intensity (2-62), the parameters of which are determined from the "adiabatic" experiments, is suitable for calculation of the temperature fields of massive concrete water engineering structures, which were constructed observing the necessary technical conditions (sufficient heat insulation of the surface at low ambient temperatures, etc.);

c) The heat liberation intensity function suggested by I. D. Zaporozhets is used,

$$q = q_{20} 2^{\frac{T-20}{10}} \left[1 + A_{20} \int_0^\tau 2^{\frac{T-20}{10}} d\tau \right]^{-1,933}, \tag{2-63}$$

where

$$q_{20} = 0.833\, Q_{max} A_{20}.$$

The numerical values of parameters Q_{max} and A_{20} are established from an experimental (or otherwise produced) curve of isothermal heat liberation of the concrete of the composition used at the standard temperature T_{CT} = 20 C; this curve is approximated by the expression

$$Q = Q_{max} \{1 + [1 + A_{20}\tau]^{-0,833}\}.$$

The methods of determination of the approximate values of Q_{max} and A_{20}, based on data on the composition of the concrete, are presented in § 2-2.

The heat exchange of concrete masses with the surrounding environment follows the linear boundary conditions of the third kind

$$\left.\frac{\partial T}{\partial n}\right|_\Gamma = h\,[\Phi(\mathcal{P}, \tau) - T|_\Gamma], \tag{2-64}$$

where n is the external normal to the boundary surface at point , $\Phi(\mathcal{T}, \tau)$ is the effective temperature of the surrounding environment; $h = \beta/\lambda$ is the relative heat transfer coefficient; β is the heat transfer coefficient

$$\frac{1}{\beta} = \frac{1}{\alpha} + \sum_{i=1}^{k} \frac{R_{d\,i}}{\lambda_i};$$

α is the heat transfer coefficient from the surface in direct contact with the environment; R_{di} and λ_i are the thickness and heat transfer conductivities of the ith layer ($i = 1, 2, \ldots, k$)[1] of the deck.

In the general case

$$\alpha = \alpha_c + \alpha_{rad},$$

where α_c and α_{rad} are the heat transfer coefficients due to convection and radiation respectively;

$$\Phi(\mathcal{P}, \tau) = \psi(\mathcal{P}, \tau) + \frac{F(\mathcal{P}, \tau)}{\beta},$$

where $\psi(\mathcal{T}, \tau)$ is the ambient temperature; $F(\mathcal{T}, \tau)$ is the heat flux from the external source (solar radiation, etc.) absorbed by the surface.

The surrounding air temperature is satisfactorily described by the sum of three simple harmonic functions with the following periods: annual, semi-monthly and daily. During the period of construction, other rules of change of air temperature are possible (temperature constant, piecewise-continuous, represented in tabular form, etc.). In areas of contact of the body with the water, due to the high values of heat transfer coefficient α, the temperature of the surface of the body (or deck) is assumed equal to the temperature of the water, i.e., boundary conditions of the first kind are assigned. The temperature of the water in the reservoir is either determined by analogy, or predicted by the method developed by A. I. Pekhovich.

The material we have just presented allows us to formulate the problem of heat conductivity for concrete structures as an edge problem for a Fourier differential equation of the following type

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + \frac{1}{c\gamma} q. \qquad (2\text{-}65)$$

The intensity of heat liberation q in the general case is a function of coordinates x_j ($j = 1, 2, 3$), time τ and temperature T. If we assign q as a function of coordinates and only time or as a generalized function of heat liberation intensity (2-62), then equation (2-65) will be linear, permitting

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[1]If there is no deck, $k = 0$.

analytic solution for bodies of classical shape (wall, cylinder, prism of rectangular and circular cross section, parallelepiped, etc.). If we use the heat liberation intensity function in the form suggested by I. D. Zaporozhets [formula (2-63)], differential equation (2-65) will be nonlinear and its solution can be produced only numerically.

In calculating the temperature fields of concrete water engineering structures, we must distinguish between the construction and operation periods. The characteristic peculiarities of the construction period from the standpoint of the formation of the temperature mode of the structure are heat liberation in the concrete mass and change in the calculated area, discrete or continuous with time, caused by the very nature of the process of erection of the structure.

The operational period refers to the period after the structure has been constructed. During this period, the temperature field of the structure is determined basically by the temperature of the surrounding air, temperature of the water in the reservoir, insolation, etc.; heat liberation in the concrete can be ignored. At the beginning of this period, in our calculations of the temperature field, we must consider the distribution of temperature in the dam at the moment of completion of construction. With the course of time, the influence of this initial thermal state gradually decreases until, of course, it becomes so negligible that it can be ignored.

The change in the temperature of the environment (air, water, etc.) is periodic in nature. Under the influence of this temperature, after a certain (rather long) time following erection, a stable periodic or quasistable thermal state develops in the dam -- the temperature at any point in the structure becomes a periodic function of time.

Thus, during the period of operation, we can note two modes in the temperature state of the structure: transient and quasistable.

The transient mode is distinguished by the dependence of the temperature field of the structure on its initial temperature state and the ambient temperature, the quasistable mode -- by the dependence of the temperature field only on the periodically changing ambient temperatures.

Of course, at any moment in time the temperature field and structure depends also on the heat-physical characteristics of the material -- the temperature conductivity coefficient, the heat conductivity coefficient and the volumetric specific heat capacity.

The problem of heat conductivity for concrete structures in the transient mode during the operational period is formulated as a problem for a Fourier equation

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T \tag{2-66}$$

with assigned initial and boundary conditions.

The problem of heat conductivity for concrete structures in the quasistable mode during the operational period is formulated as an edge problem for equation (2-66) without initial conditions.

In order to analyze the temperature field in this case, it is frequently sufficient to represent the temperature of the medium as a simple harmonic function with one annual harmonic, for example as

$$\Phi(\mathcal{T}, \tau) = T_m(\mathcal{T}) + A(\mathcal{T}) \sin(\omega\tau + \varepsilon). \qquad (2\text{-}67)$$

In (2-67), the mean annual temperature of the air T_m and amplitude A are piecewise-constant functions of the position of the point $\mathcal{T}$ on the boundary of the surface. This means that the fluctuations in temperature of the medium can be assigned only for a portion of the surface (for this portion $A(\mathcal{T}) = A = \text{const}$), while over the remainder of the surface, the temperature of the environment may be piecewise-constant [in this portion $A(\mathcal{T}) = 0$][1].

Let us write harmonic function (2-67) in complex form

$$\Phi(\tau) = T_m + Ae^{i(\omega\tau+\varepsilon)}. \qquad (2\text{-}68)$$

We assume

$$T = v + w, \qquad (2\text{-}69)$$

where function v satisfies the Laplace equation

$$\nabla^2 v = 0 \qquad (2\text{-}70)$$

and the boundary conditions

$$\frac{\partial v}{\partial n}\Big|_\Gamma = h[T_m(\mathcal{T}) - v|_\Gamma], \qquad (2\text{-}71)$$

[1]Further discussions will remain in force even in the case when in certain sectors of the surface a constant, even 0, heat flux is assigned.

while function w satisfies the Fourier equation

$$\frac{\partial w}{\partial \tau}=a\nabla^2 w \tag{2-72}$$

and the boundary conditions

$$\left.\frac{\partial w}{\partial n}\right|_{\Gamma}= h[A(\mathcal{G})e^{i(\omega\tau+\varepsilon)}-w|_{\Gamma}]. \tag{2-73}$$

The solution of problem (2-72)-(2-73) will be sought as

$$w=u(x, y, z)e^{i(\omega\tau+\varepsilon)}. \tag{2-74}$$

Then, as we can easily show, to determine function u we produce the differential equation

$$\nabla^2 u - i\frac{\omega}{a}u=0 \tag{2-75}$$

and the boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma}=h[A(\mathcal{G})-u|_{\Gamma}]. \tag{2-76}$$

In the final form, the temperature function is represented as a sum

$$T=v+\operatorname{Im} we^{i(\omega\tau+\varepsilon)}. \tag{2-77}$$

Thus, the temperature field of the structure (or its elements) in the quasistable mode of the operational period can be established as a result of solution of two problems: one, yielding values of the stable component v (problem (2-70)-(2-71)), and another, yielding values of the quasistable component w [problem (2-75)-(2-76)].

This path is recommended in those cases when the solution is produced by analytic methods.

When we use numerical methods, it is simpler to start with differential equation (2-66), corresponding to the boundary conditions and an arbitrary,

for example 0, initial condition. The solution in this case will be the temperature field established after a sufficiently long time interval.

CHAPTER 3. METHODS OF SOLUTION OF THE PROBLEM OF HEAT CONDUCTIVITY

3-1. Some Methods of Simplification of the General Problem of Heat Conductivity

Principle of Superposition

Let us study the solution to the following problem

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + Q(x_1, x_2, x_3, \tau)$$
$$(-R_1 < x_1 < R_2, \ -L_1 < x_2 < L_2, \ -D_1 < x_3 < D_2, \ \tau > 0);$$
$$T(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3)$$
$$(-R_1 \leqslant x_1 \leqslant R_2, \ -L_1 \leqslant x_2 \leqslant L_2, \ -D_1 \leqslant x_3 \leqslant D_2);$$
$$\alpha \frac{\partial T}{\partial n}\bigg|_\Gamma + \beta T|_\Gamma = \gamma_1 g_1(\mathcal{S}) + \gamma_2 g_2(\mathcal{S}, \tau).$$

Suppose

$$T(x_1, x_2, x_3, \tau) = u(x_1, x_2, x_3, \tau) + v(x_1, x_2, x_3) +$$
$$+ \omega(x_1, x_2, x_3, \tau) + w(x_1, x_2, x_3, \tau),$$

where u, v, ω and w are solutions of simpler problems namely

$$\frac{\partial u}{\partial \tau} = a\nabla^2 u + Q;$$
$$u(x_1, x_2, x_3, 0) = 0;$$
$$\alpha \frac{\partial u}{\partial n}\bigg|_\Gamma + \beta u|_\Gamma = 0;$$
$$\nabla^2 v = 0;$$
$$\alpha \frac{\partial v}{\partial n}\bigg|_\Gamma + \beta v|_\Gamma = \gamma_1 g_1(\mathcal{S});$$
$$\frac{\partial \omega}{\partial \tau} = a\nabla^2 \omega;$$

$$\omega(x_1, x_2, x_3, 0) = 0;$$
$$\alpha \frac{\partial \omega}{\partial n}\Big|_\Gamma + \beta \omega |_\Gamma = \gamma_2 g_2(\mathcal{P}, \tau);$$
$$\frac{\partial w}{\partial \tau} = a \nabla^2 w;$$
$$w(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3) - v(x_1, x_2, x_3);$$
$$\alpha \frac{\partial w}{\partial n}\Big|_\Gamma + \beta w |_\Gamma = 0.$$

The sum of functions u, v, ω and w satisfies the differential equation and edge conditions of the main problem and is therefore its unique solution.

The use of the principle of superposition results from the linearity of the differential equation and edge conditions of the problem of heat conductivity.

Multiplication of Solutions

Let us assume we must solve the two-dimensional problem

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) + \frac{\partial^2 T}{\partial \zeta^2}\right]$$
$$(R_1 < \xi < R_2,\ 0 < \zeta < L,\ \tau > 0,\ i = 0 \vee 1)^*;$$
$$T(\xi, \zeta, 0) = f(\xi, \zeta)\ (R_1 \leqslant \xi \leqslant R_2,\ 0 \leqslant \zeta \leqslant L);$$
$$\alpha_j \frac{\partial T(R_j, \zeta, \tau)}{\partial \xi} + (-1)^j \beta_j T(R_j, \zeta, \tau) = 0 \quad (j = 1,\ 2),$$
$$\alpha_k \frac{\partial T(\xi, (k-3)L, \tau)}{\partial \zeta} + (-1)^k \beta_k T(\xi, (k-3)L, \tau) = 0\ (k = 3,\ 4).$$

Suppose

$$f(\xi, \zeta) = f_1(\xi) f_2(\zeta).$$

Then, as we can easily see by direct substitution

$$T = T_1(\xi, \tau) T_2(\zeta, \tau),$$

where $T_1(\xi, \tau)$ is the solution of the one-dimensional problem

[1] The symbol $\vee$ represents "or"; re recall that ξ = x, ζ = 7, i = 0 in rectangular coordinates, ξ = r, ζ = z, i = 1 in cylindrical coordinates.

$$\frac{\partial T_1}{\partial \tau}=a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T_1}{\partial \xi}\right)\ (R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1);$$
$$T(\xi,\ 0)=f_1(\xi)\ (R_1\leqslant\xi\leqslant R_2);$$
$$\alpha_j\frac{\partial T_1(R_j,\ \tau)}{\partial \xi}+(-1)^j\beta_j T_1(R_j,\ \tau)=0\quad (j=1,\ 2),$$

while $T_2(\zeta,\ \tau)$ is the solution of the one-dimensional problem

$$\frac{\partial T_2}{\partial \tau}=a\frac{\partial^2 T_2}{\partial \zeta^2}\quad (0<\zeta<L,\ \tau>0);$$
$$T_2(\zeta,\ 0)=f_2(\zeta)\quad (0\leqslant\zeta\leqslant L);$$
$$\alpha_k\frac{\partial T_2((k-3)L,\ \tau)}{\partial \zeta}+(-1)^k\beta_k T_2((k-3)L,\ \tau)=0\ (k=3,\ 4).$$

Similar relationships may obtain for three-dimensional problems as well.

The constants α and β may be equal to 0 (not simultaneously). This means that the results produced are correct with boundary conditions of the first and third kind, and also for a particular case of boundary conditions of the second kind when $\partial T/\partial x_j = 0$ ($j = 1, 2, 3$).

Duamel Theorem

Suppose the function $\theta(\xi, \tau)$ is a solution of a problem such as

$$\frac{\partial \theta}{\partial \tau}=a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial \theta}{\partial \xi}\right)\ (R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1);$$
$$\theta(\xi,\ 0)=0\quad (R_1\leqslant\xi\leqslant R_2);$$
$$\theta(R_1,\ \tau)=1,\ \theta(R_2,\ \tau)=0$$

or

$$\frac{\partial \theta(R_1,\ \tau)}{\partial \xi}=-h_1[1-\theta(R_1,\ \tau)],$$
$$\frac{\partial \theta(R_2,\ \tau)}{\partial \xi}=-h_2\theta(R_2,\ \tau).$$

Then according to the Duamel theorem, the solution $T(\xi, \tau)$ of the problem with surface temperature ϕ or ambient temperature ψ, depending on time [$\phi = \phi(\tau)$ or $\psi = \psi(\tau)$] and 0 initial temperature, i.e., the solution of the problem

$$\frac{\partial T}{\partial \tau}=a\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial T}{\partial \xi}\right)(R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1);$$
$$T(\xi,\ 0)=0\ (R_1\leqslant\xi\leqslant R_2);$$
$$T(R_1,\ \tau)=\varphi(\tau),\ T(R_2,\tau)=0$$

or

$$\frac{\partial T(R_1,\ \tau)}{\partial \xi}=-h_1[\psi(\tau)-T(R_1,\ \tau)],$$
$$\frac{\partial T(R_2,\ \tau)}{\partial \xi}=-h_2T(R_2,\ \tau)$$

is the integral

$$T(\xi,\ \tau)=\int_0^{\tau}\varphi(\lambda)\frac{\partial}{\partial \tau}\theta(\xi,\ \tau-\lambda)\,d\lambda$$

or

$$T(\xi,\ \tau)=\int_0^{\tau}\psi(\lambda)\frac{\partial}{\partial \tau}\theta(\xi,\ \tau-\lambda)\,d\lambda.$$

The Duamel theorem is correct with all combinations of boundary conditions of the first and third kind, and also in those cases when at one end of the interval $[R_1,\ R_2]$ a homogeneous boundary condition of the second kind is assigned ($\partial T/\partial\xi = 0$). This extends to two-dimensional and three-dimensional problems with 0 initial and the corresponding boundary conditions.

Certain Special Cases

1. Suppose $T(x_1,\ x_2,\ x_3,\ \tau)$ satisfies the differential equation

$$\frac{\partial T}{\partial \tau}=a\nabla^2 T+b(\tau)T$$

and generally heterogeneous edge conditions.

Then

$$T = v(x_1, x_2, x_3, \tau) \exp\left[\int_0^\tau b(\tau)\, d\tau\right],$$

where the function $v(x_1, x_2, x_3, \tau)$ satisfies the equation

$$\frac{\partial v}{\partial \tau} = a\nabla^2 v$$

and the correspondingly changed edge conditions.

2. The differential equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + b\frac{\partial T}{\partial x} + c\frac{\partial T}{\partial y} + d\frac{\partial T}{\partial z} + gT$$

(where b, c, d, g are constants) by substitution

$$T = v(x, y, z, \tau) \exp\left[-\frac{b}{2a}x - \frac{c}{2a}y - \frac{d}{2a}z - \right.$$
$$\left. - \frac{b^2}{4a}\tau - \frac{c^2}{4a}\tau - \frac{d^2}{4a}\tau + g\tau\right]$$

is converted to the form

$$\frac{\partial v}{\partial \tau} = a\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right).$$

3-2. Method of Separation of Variables (Fourier Method)

Let us assume that we must find the temperature function $T(\xi, \tau)$ satisfying the differential equation

$$\frac{\partial T}{\partial \tau} = a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) \quad (R_1 < \xi < R_2,\ \tau > 0,\ i = 0 \vee 1), \tag{3-1}$$

the initial condition

$$T(\xi, 0) = f(\xi) \quad (R_1 \leqslant \xi \leqslant R_2) \tag{3-2}$$

and boundary conditions of the general form

$$\alpha_j \frac{\partial T(R_j, \tau)}{\partial \xi} + (-1)^j \beta_j T(R_j, \tau) = 0 \quad (j = 1, 2). \tag{3-3}$$

We assume

$$T = U(\xi) V(\tau). \tag{3-4}$$

Let us substitute relationship (3-4) into equation (3-1) and divide the expression produced by UV.

We have:

$$\frac{1}{\xi^i U(\xi)} \frac{d}{d\xi} \left[\xi^i \frac{dU(\xi)}{d\xi} \right] = \frac{1}{a} \frac{1}{V(\tau)} \frac{dV(\tau)}{d\tau}. \tag{3-5}$$

The terms in the left portion of equation (3-5) are functions of only ξ, whereas the terms in the right portion are functions only of τ. Equation (3-5) should be fulfilled with any ξ and τ. This is possible only when the left and right portions of the equation are equal to constant C.

Consequently

$$\frac{1}{a} \frac{1}{V(\tau)} \frac{dV(\tau)}{d\tau} = C;$$

$$\frac{1}{\xi^i U(\xi)} \frac{d}{d\xi} \left[\xi^i \frac{dU(\xi)}{d\xi} \right] = C. \tag{3-6}$$

With an accuracy equal to the integration constant, the solution of equation (3-6) is

$$V(\tau) = e^{Ca\tau}.$$

Since the temperature field approaches the equilibrium state, it is necessary that constant C by a real negative number. This is a physical interpretation of the problem. We can show that where $C \geq 0$, function $U(\xi)$ is identical to 0 $U(\xi) \equiv 0$ and condition $C < 0$ is the only possible nontrivial, i.e., not equal to 0, solution. We can therefore assume

$$C = -\frac{\mu^2}{R^2},$$

where μ^2 is a constant; R is the characteristic dimension of the body.

In order to define function $U(\xi)$, we use the differential equation

$$\frac{1}{\xi^i}\frac{d}{d\xi}\left[\xi^i\frac{dU(\xi)}{d\xi}\right]+\frac{\mu^2}{R^2}U(\xi)=0 \quad (R_1<\xi<R_2,\ i=0\vee 1) \tag{3-7}$$

and the homogeneous boundary conditions

$$\alpha_j\frac{dU(R_j)}{d\xi}+(-1)^j\beta_j U(R_j)=0 \quad (j=1,\ 2). \tag{3-8}$$

The boundary conditions (3-8) are produced from the boundary conditions (3-3) of the initial problem by an obvious means after performing substitution (3-4).

Problem (3-7)-(3-8) is a particular case of the more general Shturm-Liouville, the basic statements of which are analyzed in various monographs and textbooks on mathematical physics (for example, see [60, 61, 86, 122, 123]).

The general integral of equation (3-7) is

$$U\left(\mu\frac{\xi}{R}\right)=AW_1\left(\mu\frac{\xi}{R}\right)+BW_2\left(\mu\frac{\xi}{R}\right),$$

where $W_k(\mu - \frac{\xi}{R})$ $(k = 1, 2)$ are two linearly independent solutions of equation (3-7); A and B are arbitrary constants.

Functions $\phi_1(\xi),\ldots,\phi_m(\xi)$ are called linearly dependent, if they satisfy precisely relative to ξ the homogeneous linear relationship

$$\sum_{k=1}^{\infty} C_k\varphi_k(\xi)=0$$

with constant coefficients C_k $(k = 1,\ldots, m)$ which are not always equal to 0. Otherwise, all of these functions are called linearly dependent.

As linearly independent solutions of equation (3-7), we accept: in rectangular coordinates -- the trigonometric functions cos μ (x/R) and sin μ (x/R), in cylindrical coordinates -- the Bessel functions of the first and second kind of zero order $J_0(\mu(r/R))$ and $Y_o(\mu(r/R))$.

Satisfying the boundary conditions (3-8), we produce the following system of two equations for constants A and B:

$$A\left[\alpha_1 \frac{d}{d\xi} W_1\left(\mu \frac{R_1}{R}\right) - \beta_1 W_1\left(\mu \frac{R_1}{R}\right)\right] + B\left[\alpha_1 \frac{d}{d\xi} W_2\left(\mu \frac{R_1}{R}\right) - \right.$$
$$\left. - \beta_1 W_2\left(\mu \frac{R_1}{R}\right)\right] = 0;$$

$$A\left[\alpha_2 \frac{d}{d\xi} W_1\left(\mu \frac{R_2}{R}\right) + \beta_2 W_1\left(\mu \frac{R_2}{R}\right)\right] +$$
$$+ B\left[\alpha_2 \frac{d}{d\xi} W_2\left(\mu \frac{R_2}{R}\right) + \beta_2 W_2\left(\mu \frac{R_2}{R}\right)\right] = 0. \tag{3-9}$$

The system of homogeneous equations (3-9) allows nontrivial[1] solutions if it is defined equal to 0, i.e.,

$$\begin{vmatrix} \alpha_1 \frac{d}{d\xi} W_1\left(\mu \frac{R_1}{R}\right) - \beta_1 W_1\left(\mu \frac{R_1}{R}\right); & \alpha_1 \frac{d}{d\xi} W_2\left(\mu \frac{R_1}{R}\right) - \beta_1 W_2\left(\mu \frac{R_1}{R}\right) \\ \alpha_2 \frac{d}{d\xi} W_1\left(\mu \frac{R_2}{R}\right) + \beta_2 W_1\left(\mu \frac{R_2}{R}\right); & \alpha_2 \frac{d}{d\xi} W_2\left(\mu \frac{R_2}{R}\right) + \beta_2 W_2\left(\mu \frac{R_2}{R}\right) \end{vmatrix} = 0. \tag{3-10}$$

Condition (3-10) is fulfilled with certain discrete values of the parameter $\lambda_n^2 = \frac{\mu_n^2}{R^2}$. The values of λ_n^2 yielding nontrivial solutions of problem (3-7)-(3-8) are called Eigenvalues (numbers), while the corresponding nontrivial solutions are called Eigenfunctions of the problem.

Equation (3-10) will be called the characteristic equation of the problem, numbers u_n are its roots.

The expressions for the Eigenfunctions are established with accuracy equivalent to a constant factor.

From the first equation of system (3-9) we have

---

[1]Not identical to zero.

$$B = -\frac{\alpha_1 \frac{d}{d\xi} W_1\left(\mu_n \frac{R_1}{R}\right) - \beta_1 W_1\left(\mu_n \frac{R_1}{R}\right)}{\alpha_1 \frac{d}{d\xi} W_2\left(\mu_n \frac{R_1}{R}\right) - \beta_1 W_2\left(\mu_n \frac{R_1}{R}\right)} A.$$

Then from the second equation of system (3-9) it follows that the Eigenfunction of the problem $U_0(\mu_n(\xi/R))$ belonging to the Eigenvalue λ_n can be taken, for example, as

$$U_0\left(\mu_n \frac{\xi}{R}\right) = \left[\alpha_1 \frac{d}{d\xi} W_2\left(\mu_n \frac{R_1}{R}\right) - \beta_1 W_2\left(\mu_n \frac{R_1}{R}\right)\right]\left[\alpha_2 \frac{d}{d\xi} W_1\left(\mu_n \frac{R_2}{R}\right) + \beta_2 W_1\left(\mu_n \frac{R_2}{R}\right)\right] W_1\left(\mu_n \frac{\xi}{R}\right) - \left[\alpha_1 \frac{d}{d\xi} W_1\left(\mu_n \frac{R_1}{R}\right) - \beta_1 W_1\left(\mu_n \frac{R_1}{R}\right)\right]\left[\alpha_2 \frac{d}{d\xi} W_2\left(\mu_n \frac{R_2}{R}\right) + \beta_2 W_2\left(\mu_n \frac{R_2}{R}\right)\right] W_2\left(\mu_n \frac{\xi}{R}\right).$$

In the Shturm-Liouville theory it is proven that the Eigenvalues form an increasing sequence of real positive numbers, so that

$$\lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \ldots$$

Each Eigenvalue $\lambda_n^2 = \frac{\mu_n^2}{R^2}$ $(n = 1, 2, \ldots, \infty)$ corresponds to but one, with an accuracy to a constant factor, Eigenfunction $U_0(\mu_n \frac{\xi}{R})$.

The Eigenfunctions corresponding to various Eigenvalues are orthogonal with weight ξ^i, i.e.

$$\int_{R_1}^{R_2} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) U_0\left(\mu_m \frac{\xi}{R}\right) d\xi = 0 \quad (n \neq m). \tag{3-11}$$

Let us return to solution of problem (3-1)-(3-3).

The expression

$$T_n = a_n U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}},$$

where a_n is a constant is a particular solution of equation (3-1), satisfying the boundary conditions (3-3).

Since the differential equation (3-1) is linear, any finite number of its partial solutions is also a solution. This leads us to the thought of construction of a solution of problem (3-1)-(3-3) in the form of an infinite series

$$T=\sum_{n=1}^{\infty} T_n=\sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

From this, as $\tau \to 0$

$$T(\xi, 0)=f(\xi)=\sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{\xi}{R}\right). \tag{3-12}$$

In order to determine the coefficients a_n, both parts of equation (3-12) are multiplied by $\xi^i U_0(\mu_m \frac{\xi}{R})$ and integrated within limits of R_1 to R_2. Due to the orthogonality of the Eigenfunctions [see orthogonality relationship (3-11)], all of the integrals to the right, with the exception of one, for which n = m, are equal to zero.

Consequently

$$\int_{R_1}^{R_2} \xi^i f(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = a_n \int_{R_1}^{R_2} \xi^i U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$$

and

$$a_n=\frac{1}{\|U_0\|^2}\int_{R_1}^{R_2} \xi^i f(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi, \tag{3-13}$$

where $\|U_0\|^2=\int_{R_1}^{R_2} \xi^i U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$ is the square of the norm of the system of Eigenfunctions $\left\{U_0\left(\mu_n \frac{\xi}{R}\right)\right\}$.

The coefficients a_n, defined by formula (3-13), are called Fourier coefficients, while series (3-12) is the Fourier series generalization of function $f(\xi)$. Thus,

$$T(\xi, \tau) = \sum_{n=1}^{\infty} \frac{U_0\left(\mu_n \frac{\xi}{R}\right)}{\|U_0\|^2} \int_{R_1}^{R_2} \xi^j f(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi e^{-\mu_n^2 \frac{a\tau}{R^2}}. \tag{3-14}$$

It can be shown that the solution produced (3-14) is the classical solution if the initial function $f(\xi)$ is continuous and matched with the boundary conditions.

As was shown in § 1-2, the introduction of the concept of the generalized solution softens these requirements. Formula (3-14) yields the generalized solution of the problem if function $f(\xi)$ is piecewise-continuous in its main area in which it is fixed, and is not matched with the boundary conditions.

Earlier, we produced a solution to the problem of heat conductivity for a finite body with homogeneous differential equation and homogeneous boundary conditions. It is to problems of this type that we can directly apply the method of separation of variables (Fourier method). As concerns problems with heterogeneous equations and heterogeneous boundary conditions, in this case we must first, using various artificial methods, reduce them to homogeneous, then begin solution by the Fourier method. It should be noted that the solutions produced by the Fourier method are not always convenient for numerical realization, since they frequently contain slowly converging series.

Example. Find the temperature field of an unlimited concrete wall with an initial temperature which varies through its thickness. Heat exchange between the wall and the surrounding environment is by the rule of convection (boundary conditions of the third kind) with various values of the relative heat transfer factor h_j ($j = 1, 2$) at the boundary surfaces. The ambient temperature is zero.

An unlimited wall, or simply a wall, refers to a body, two dimensions of which (for example height and length) are significantly greater than the third (width).

Since the assigned initial temperature is a function of coordinate x through the width of the wall, then

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0.$$

and the problem which we are studying is one-dimensional.

Therefore, this problem is formulated mathematically as follows: we must find the temperature function T(x, τ) satisfying the differential equation

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}\ (0 < x < R, \tau > 0). \tag{3-15}$$

the initial condition

$$T(x, 0) = f(x)\ (0 \leqslant x \leqslant R) \tag{3-16}$$

and the boundary conditions

$$\frac{\partial T((j-1)R, \tau)}{\partial x} = (-1)^{j+1} h_j T((j-1)R, \tau)\ (j = 1, 2). \tag{3-17}$$

Let us write a particular solution[1]

$$T_n = a_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}\ (n = 1, 2\ldots),$$

where $U_0(\mu_n \frac{x}{R})$ is the Eigenfunction of the problem; μ_n^2/R^2 are the Eigenvalues; a_n is a constant.

The Eigenfunction $U_0(\mu_n \frac{x}{R})$ is the solution of the corresponding Shturm-Liouville problem

$$\frac{d^2 U_0}{dx^2} + \frac{\mu_n^2}{R^2} U_0 = 0\ (0 < x < R); \tag{3-18}$$

$$\frac{dU_0((j-1)R)}{dx} + (-1)^j h_j U_0((j-1)R) = 0\ (j = 1, 2). \tag{3-19}$$

The common integral of differential equation (3-18) is:

[1]It is produced obviously after separation of variables x and τ.

$$U_0\left(\mu_n \frac{x}{R}\right) = B_n \cos \mu_n \frac{x}{R} + C_n \sin \mu_n \frac{x}{R}.$$

Constants B_n and C_n are defined from boundary conditions (3-19), which, after substitution of the last expression into them, are converted to a homogeneous system of equations

$$\mathrm{Bi}_1 B_n - \mu_n C_n = 0;$$
$$(\mathrm{Bi}_2 \cos \mu_n - \mu_n \sin \mu_n) B_n + (\mu_n \cos \mu_n + \mathrm{Bi}_2 \sin \mu_n) C_n = 0,$$

where $\mathrm{Bi}_j = h_j R = \frac{a_j R}{\lambda}$ ($j = 1, 2$) is the Biot criterion.

In order for this system of homogeneous equations to have non-zero solutions, its determinant must be equal to 0, i.e.,

$$\begin{vmatrix} \mathrm{Bi}_1 & -\mu_n \\ \mathrm{Bi}_2 \cos \mu_n - \mu_n \sin \mu_n & \mu_n \cos \mu_n + \mathrm{Bi}_2 \sin \mu_n \end{vmatrix} = 0.$$

From this we produce the characteristic equation of the problem

$$\cot \mu_n = \frac{\mu_n^2 - \mathrm{Bi}_1 \mathrm{Bi}_2}{\mu_n (\mathrm{Bi}_1 + \mathrm{Bi}_2)} \quad (n = 1, 2\ldots), \tag{3-20}$$

the roots of which μ_n yield the spectrum of Eigennumbers $\lambda_n^2 = \frac{\mu_n^2}{R^2}$

From the first equation of the system of homogeneous equations it follows that:

$$C_n = \frac{\mathrm{Bi}_1}{\mu_n} B_n.$$

Therefore, with an accuracy to a constant factor, the Eigenfunction is

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R}. \tag{3-21}$$

The general solution is written in the form

$$T(x,\tau)=\sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Satisfying initial condition (3-17), we produce:

$$f(x)=\sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{x}{R}\right). \tag{3-22}$$

Expression (3-22) is an expansion of function f(x) into a generalized series with respect to the Eigenfunctions of the Shturm-Liouville problem $U_0(\mu_n\frac{x}{R})$.

In order to determine the coefficients of expansion a_n, we multiply expression (3-22) by $U_0(\mu_m\frac{x}{R})$ and integrate from 0 to R. Considering the orthogonality of the system of functions $\{U_0(\mu_n\frac{x}{R})\}$, we find:

$$a_n=\frac{1}{\|U_0\|^2}\int_0^R f(x)\, U_0\left(\mu_n \frac{x}{R}\right) dx,$$

where $\|U_0\|^2=\int_0^R U_0^2\left(\mu_n \frac{x}{R}\right) dx$ is the square of the norm of the Eigenfunction.

It can be shown that $||U_0||^2$ is equal to:

$$\|U_0\|^2=\int_0^R\left(\mu_n \cos \mu_n \frac{x}{R}+\mathrm{Bi}_1 \sin \mu_n \frac{x}{R}\right)^2 dx=$$
$$=\frac{R}{2}\left[\mu_n^2+\mathrm{Bi}_1+\mathrm{Bi}_2+\frac{\mathrm{Bi}_2(\mu_n^2+\mathrm{Bi}_1^2)}{\mu_n^2+\mathrm{Bi}_2^2}\right]. \tag{3-23}$$

Thus, the general solution of the problem in question (3-15)-(3-17) is

$$T(x,\tau)=\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\int_0^R f(x)\, U_0\left(\mu_n \frac{x}{R}\right) dx\, U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \mathrm{Fo}}, \tag{3-24}$$

where μ_n is the root of characteristic equation (3-20); $F_o = a\tau/R^2$ is the Fourier criterion; $U_0(\mu_n\frac{x}{R})$ and $||U_0||^2$ are the Eigenfunction and the square of its norm, defined by formulas (3-21) and (3-23) respectively.

If the initial temperature is constant, i.e., $f(x) = T_0$, then

$$T = T_0 \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \mathrm{Fo}},$$

where

$$A_n = \frac{N_n}{\|U_0\|^2};$$

$$N_n = \int_0^R U_0\left(\mu_n \frac{x}{R}\right) dx = \frac{R}{\mu_n}\left[\mathrm{Bi}_1 + (-1)^{n+1}\mathrm{Bi}_2 \sqrt{\frac{\mu_n^2 + \mathrm{Bi}_1^2}{\mu_n^2 + \mathrm{Bi}_2^2}}\right].$$

3-3. Finite Integral Transforms of G. A. Greenberg

Direct and Reverse Transforms

The integral transform of function $U(\eta)$ with respect to variable η refers to an integral of the following type

$$\bar{u}(\omega) = \int_{R_1}^{R_2} u(\eta) K(\omega, \eta)\, d\eta.$$

This transform of function $u(\eta)$ (original) sets it in correspondence with function $\bar{u}(\omega)$ (the transform or mapping). Function $K(\omega, \eta)$ is the kernel of the integral transform. The transform can be performed both with respect to the spatial variable ξ, and to the time variable τ.

A distinction is drawn between finite integral transforms, when the interval of integration is finite $[R_1, R_2]$ and infinite integral transforms, when the integration interval is semiinfinite $(-\infty, R_2]$ and $[R_1, \infty)$ or infinite $(-\infty, \infty)$.

The expression with which the mapping $\bar{u}(\omega)$ can be used to restore the original $U(\eta)$ is called the reverse transform or inversion formula.

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In addition to homogeneous integral transforms of this type, the theory of heat conductivity frequently utilizes multidimensional integral transforms, i.e., transforms with respect to n variables $\eta_1, \eta_2, \ldots, \eta_n$, both temporal and spatial. These transforms are defined by expressions such as

$$\overline{u}(\omega_1, \ldots, \omega_n) = \overbrace{\iint\limits_{(S)} \cdots \int}^{n} u(\eta_1, \ldots, \eta_n) K(\omega_1, \ldots, \omega_n; \eta_1, \ldots, \eta_n) d\eta_1 \ldots d\eta_n.$$

Here $K(\omega_1, \ldots, \omega_n; \eta_1, \ldots, \eta_n)$ is the kernel of the integral transform; S is the area of integration.

In the theory of heat conductivity, we utilize various types of integral transforms -- Laplace transforms, finite G. A. Greenberg transforms, Fourier transforms, Hankel transforms, Mellin transforms, etc. The theory of integral transforms and examples of their utilization in mathematical physics have been published in original works, monographs and handbooks (for example see [33, 34, 45, 54, 60, 68, 116, 126]).

In this book, primary attention is given to finite integral G. A. Greenberg transforms (§ 3-3) and Laplace transforms (§ 3-4). These transforms are used to solve various problems of heat conductivity for concrete bodies.

The finite integral G. A. Greenberg transforms are effective in the solution of problems of heat conductivity for finite bodies, when the differential equation and boundary conditions are heterogeneous, while the initial conditions may be of any type. In this case, the coefficients before the temperature function or its derivatives must be constant or, possible, functions of time and coordinates with respect to which transformation is not conducted. Furthermore, the differential equations should not contain a first deriviative with respect to the coordinate of transformation.

In the rectangular and cylindrical (with axis of symmetry) systems of coordinates[1], the G. A. Greenberg finite integral transform of function $u(\xi)$ with respect to variable ξ over the interval $[R_1, R_2]$ is defined by the formula

$$\overline{u}_n = \int\limits_{R_1}^{R_2} \xi^i u(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi \quad (i = 0 \vee 1). \tag{3-25}$$

[1]Let us recall: in the rectangular system of coordinates $\xi = x$, $i = 0$, in the cylindrical system of coordinates $\xi = r$, $i = 1$.

The main portion of the kernel of integral transform (3-25) is the Eigenfunction $U_0(\mu_n(\xi/R))$, belonging to the Eigenvalue of the corresponding Shturm-Liouville problem. In formula (3-25), in addition to this, ξ^i is the weight function, μ_n is the root of the characteristic equation.

One sufficient condition for existence of the transform (mapping) $\bar{u}_n$ is piecewise continuity of function $u(\xi)$ in the interval $[R_1, R_2]$.

In the construction of a reverse transform or inversion formula, we turn our attention to the following theoretical statements.

If function $u(\xi)$ is quadratically integrable in interval $[R_1, R_2]$, i.e., if there is an integral

$$\int_{R_1}^{R_2} \xi^i [u(\xi)]^2 d\xi > 0,$$

then the generalized Fourier series of this function

$$u(\xi) = \sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{\xi}{R}\right),$$

where a_n are the generalized Fourier coefficients, equal to

$$a_n = \frac{1}{\|U_0\|^2} \int_{R_1}^{R_2} \xi^i u(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$

$$\|U_0\|^2 = \int_{R_1}^{R_2} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi,$$

converges to it on the average, i.e., in the sense of the mean square, and can be integrated term by term.

If the function $u(\xi)$ is piecewise-continuous over the interval $[R_1, R_2]$ and has a quadratically integrable derivative, the generalized Fourier series

$$u(\xi) = \sum_{n=1}^{\infty} a_n U_0\left(\mu_n \frac{\xi}{R}\right),$$

converges to it absolutely and evenly in all particular areas of the main interval, not containing points of discontinuity of the function, while at discontinuity points the sum of this series is equal to the arithmetic mean of the right and left limiting values of the function.

As was noted earlier, all of these conditions are satisfied by the functions which we encounter in the book.

Thus, we can write

$$u(\xi)=\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)\int_{R_1}^{R_2}\xi^i u(\xi)U_0\left(\mu_n\frac{\xi}{R}\right)d\xi,$$

or, considering the formula for the direct transform (3-25),

$$u(\xi)=\sum_{n=1}^{\infty}\frac{\bar{u}_n}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right). \tag{3-26}$$

Expression (3-26) is the reverse transform or inversion formula.

Transform $\bar{u}_n$, with an accuracy to the square of the norm of the Eigenfunction $||U_0||^2$, agrees with the Fourier coefficient of expansion of function $u(\xi)$ into a generalized Fourier series with respect to the Eigenfunctions of the Shturm-Liouville problem. However, we know that the piecewise-continuous function is unambiguously defined by its Fourier coefficients. Consequently, if as a result of any permissible operations we establish the transform (mapping) $\bar{u}_n$, the desired function itself (original) $u(\xi)$ is uniquely defined by the inversion formula (3-26).

In inversion formula (3-26), summation is conducted with respect to all Eigenvalues (or the corresponding roots of the characteristic equation) of the Shturm-Liouville problem. The Eigenvalues form an increasing sequence of positive numbers. Only with boundary conditions of the second kind at both ends of interval $[R_1, R_2]$ is there an Eigenvalue equal to 0, which has a constant Eigenfunction. In all other cases, the spectra in the Shturm-Liouville problem contain no zeros. Therefore, the inversion formula using Eigenvalues from the Shturm-Liouville problem with boundary conditions of the second kind at both ends of the interval $[R_1, R_2]$ is written as follows:

$$u(\xi)=\lim_{\mu_n\to 0}\frac{\bar{a}_n U_0\left(\mu_n\frac{\xi}{R}\right)}{\|U_0\|^2}+\sum_{n=1}^{\infty}\frac{\bar{a}_n}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right). \qquad (3\text{-}27)$$

It can be shown that the free term in the inversion formula (3-27) containing the Eigenfunctions of the Shturm-Liouville problem of the form[1]

$$\frac{d}{d\xi}\left[\xi^i\frac{dU_0}{d\xi}\right]+\frac{\mu_n^2}{R^2}\xi^i U_0=0\ (R_1<\xi<R_2,\ i=0\vee 1);$$
$$\frac{dU_0(R_j)}{d\xi}=0\ (j=1,\ 2),$$

is the mean integral value of function $u(\xi)$, i.e.,

$$\lim_{\mu_n\to 0}\frac{\bar{a}_n U_0\left(\mu_n\frac{\xi}{R}\right)}{\|U_0\|^2}=\frac{\int\limits_{R_1}^{R_2}\xi^i u(\xi)\,d\xi}{\int\limits_{R_1}^{R_2}\xi^i d\xi}.$$

Thus, in addition to (3-27), the inversion formula with boundary conditions of the second kind can be represented as

$$u(\xi)=\frac{\int\limits_{R_1}^{R_2}\xi^i u\,d\xi}{\int\limits_{R_1}^{R_2}\xi^i d\xi}+\sum_{n=1}^{\infty}\frac{\bar{a}_n}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right). \qquad (3\text{-}28)$$

[1]We reach this problem of Eigenvalues, for example, by solution of the heat conductivity problem

$$\frac{\partial T}{\partial\tau}=a\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial T}{\partial\xi}\right)+p(\tau)T+Q(\xi,\ \tau)\ (R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1);$$
$$T(\xi,\ 0)=f(\xi)\ (R_1\leqslant\xi\leqslant R_2);$$
$$\frac{\partial T(R_j,\ \tau)}{\partial\xi}=(-1)^j\frac{\eta_j(\tau)}{\lambda}\ (j=1,\ 2).$$

Use of the G. A. Greenberg transforms for solution of one-dimensional problems.

Let us assume that we must solve the following problem:

$$\frac{\partial T}{\partial \tau}=a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right)+p(\tau)T+Q(\xi, \tau)\ (R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1); \tag{3-29}$$

$$T(\xi, 0)=f(\xi)\ (R_1\leqslant\xi\leqslant R_2); \tag{3-30}$$

$$\frac{\partial T(R_j, \tau)}{\partial \xi}=(-1)^j h_j[\psi_j(\tau)-T(R_j, \tau)]\ (j=1, 2). \tag{3-31}$$ [1]

The corresponding Shturm-Liouville problem, which can be produced after separation of variables ξ and τ, is:

$$\frac{d}{d\xi}\left[\xi^i\frac{dU_0(\lambda_n\xi)}{d\xi}\right]+\lambda_n^2\xi^iU_0(\lambda_n\xi)=0\ (R_1<\xi<R_2,\ i=0\vee 1); \tag{3-32}$$

$$\frac{dU_0(\lambda_nR_j)}{d\xi}+(-1)^j h_jU_0(\lambda_nR_j)=0. \tag{3-33}$$

The solution of the Shturm-Liouville problem (3-32)-(3-33) is the Eigenfunction $U_0(\lambda_n\xi)$, defined with an accuracy equivalent to a constant factor, belonging to the Eigenvalue λ_n. Let us assume

$$\lambda_n^2=\frac{\mu_n^2}{R^2},$$

where R is the characteristic dimension of the body, and write the Eigenfunction as

$$U_0(\lambda_n\xi)=U_0\left(\mu_n\frac{\xi}{R}\right).$$

Then, in solving problem (3-29)-(3-31), the finite integral transform should be defined by the formulas

$$\bar{T}_n(\tau)=\int_{R_1}^{R_2}\xi^iT(\xi, \tau)U_0\left(\mu_n\frac{\xi}{R}\right)d\xi; \tag{3-34}$$

[1] Boundary conditions of the third kind assigned for more definition.

$$T(\xi, \tau) = \sum_{n=1}^{\infty} \frac{\overline{T}_n(\tau)}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right), \tag{3-35}$$

where

$$\|U_0\|^2 = \int_{R_1}^{R_2} \xi^i U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi.$$

In order to seek out the transform $\overline{T}_n$, the right and left portions of differential equation (3-29) and initial condition (3-30) are multiplied by the kernel of the integral transform $\xi^i U_0(\mu_n(\zeta/R))$ and integrated between R_1 and R_2. This yields:

$$\int_{R_1}^{R_2} \frac{\partial T}{\partial \tau} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = a \int_{R_1}^{R_2} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi + \\ + p(\tau) \int_{R_1}^{R_2} \xi^i T U_0\left(\mu_n \frac{\xi}{R}\right) d\xi + \int_{R_1}^{R_2} \xi^i Q(\xi, \tau) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi. \tag{3-36}$$

The first integral in the right portion of expression (3-36) can be twice taken by parts

$$I = \int_{R_1}^{R_2} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \xi^i \frac{\partial T}{\partial \xi} U_0\left(\mu_n \frac{\xi}{R}\right)\Big|_{R_1}^{R_2} - \\ - \int_{R_1}^{R_2} \xi^i \frac{\partial T}{\partial \xi} \frac{d}{d\xi} U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \xi^i \frac{\partial T}{\partial \xi} U_0\left(\mu_n \frac{\xi}{R}\right)\Big|_{R_1}^{R_2} - \\ - \xi^i T \frac{d}{d\xi} U_0\left(\mu_n \frac{\xi}{R}\right)\Big|_{R_1}^{R_2} + \int_{R_1}^{R_2} T \frac{d}{d\xi}\left[\xi^i \frac{dU_0\left(\mu_n \frac{\xi}{R}\right)}{d\xi}\right] d\xi.$$

However, in accordance with equation (3-32) and the boundary conditions (3-31) and (3-33)

$$\frac{d}{d\xi}\left[\xi^i \frac{d}{d\xi} U_0\left(\mu_n \frac{\xi}{R}\right)\right] = -\frac{\mu_n^2}{R^2}\xi^i U_0\left(\mu_n \frac{\xi}{R}\right);$$

$$\frac{dU_0\left(\mu_n \frac{R_j}{R}\right)}{d\xi} = (-1)^{j+1} h_j U_0\left(\mu_n \frac{R_j}{R}\right) \ (j = 1,\ 2);$$

$$\frac{dT(R_j,\ \tau)}{d\xi} + (-1)^j h_j T(R_j,\ \tau) = (-1)^j h_j \psi_j(\tau) \ (j = 1,\ 2).$$

Therefore

$$I = R_2^i \frac{\partial T(R_2,\ \tau)}{\partial\xi} U_0\left(\mu_n \frac{R_2}{R}\right) - R_1^i \frac{\partial T(R_1,\ \tau)}{\partial\xi} U_0\left(\mu_n \frac{R_1}{R}\right) -$$

$$- R_2^i T(R_2,\ \tau) \frac{d}{d\xi} U_0\left(\mu_n \frac{R_2}{R}\right) + R_1^i T(R_1,\ \tau) \frac{d}{d\xi} U_0\left(\mu_n \frac{R_1}{R}\right) -$$

$$- \frac{\mu_n^2}{R^2} \int_{R_1}^{R_2} \xi^i T U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = R_2^i h_2 \psi_2(\tau) U_0\left(\mu_n \frac{R_2}{R}\right) +$$

$$+ R_1^i h_1 \psi_1(\tau) U_0\left(\mu_n \frac{R_1}{R}\right) - \frac{\mu_n^2}{R^2} \bar{T}_n.$$

Here and in the following we will assume $R_1^i|_{R_1 = 0,\ i = 0} = 1$.

Let us change the sequence of differentiation with respect to time and integration with respect to the coordinate in the left portion of expression (3-36)[1].

Then the differential equation (3-29) and the initial condition (3-30) transformed after G. A. Greenberg, become

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[1] This is possible because

$$\frac{\partial}{\partial\tau} \int_{R_1}^{R_2} \xi^i T U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \int_{R_1}^{R_2} \frac{\partial T}{\partial\tau} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi +$$

$$+ \int_{R_1}^{R_2} T \frac{\partial}{\partial\tau} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \int_{R_1}^{R_2} \frac{\partial T}{\partial\tau} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi.$$

$$\frac{d\overline{T}_n}{d\tau}=\left(-\mu_n^2\frac{a}{R^2}+p(\tau)\right)\overline{T}_n+\overline{Q}_n(\tau)+R_2^i h_2 a\psi_2(\tau)U_0\left(\mu_n\frac{R_2}{R}\right)+ \tag{3-37}$$

$$+R_1^i h_1 a\psi_1(\tau)U_0\left(\mu_n\frac{R_1}{R}\right);$$

$$\overline{T}_n(0)=\overline{f}_n, \tag{3-38}$$

where

$$\overline{Q}_n(\tau)=\int_{R_1}^{R_2}\xi^i Q(\xi,\ \tau)U_0\left(\mu_n\frac{\xi}{R}\right)d\xi;$$

$$\overline{f}_n(\tau)=\int_{R_1}^{R_2}\xi^i f(\xi)U_0\left(\mu_n\frac{\xi}{R}\right)d\xi.$$

The solution of the ordinary first order differential equation (3-37) with initial condition (3-38) is equal to:

$$\overline{T}_n=\overline{f}_n\exp\left[-\mu_n^2\frac{a\tau}{R^2}+\int_0^\tau p(\tau)\,d\tau\right]+R_2^i h_2 aU_0\left(\mu_n\frac{R_2}{R}\right)\times$$

$$\times\exp\left[-\mu_n^2\frac{a\tau}{R^2}+\int_0^\tau p(\tau)\,d\tau\right]\int_0^\tau\psi_2(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+R_1^i\ h_1 aU_0\left(\mu_n\frac{R_1}{R}\right)\exp\left[-\mu_n^2\frac{a\tau}{R^2}+\int_0^\tau p(\tau)\,d\tau\right]\times$$

$$\times\int_0^\tau\psi_1(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\int_0^\tau p(\tau)\,d\tau\right]d\tau+\exp\left[-\mu_n^2\frac{a\tau}{R^2}+\right.$$

$$\left.+\int_0^\tau p(\tau)\,d\tau\right]\int_0^\tau\overline{Q}_n(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\int_0^\tau p(\tau)\,d\tau\right]d\tau.$$

Based on the inversion formula (3-35), let us find the common solution of the problem as stated

$$T(\xi,\tau)=\exp\left[\int_0^\tau p(\tau)\,d\tau\right]\sum_{n=1}^{\infty}\left\{\frac{1}{\|U_0\|^2}\left[\left[\bar{f}_n+R_2^i h_2 a U_0\left(\mu_n\frac{R_2}{R}\right)\times\right.\right.\right.$$

$$\times\int_0^\tau \psi_2(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\int_0^\tau p(\tau)\,d\tau\right]d\tau+R_1^i h_1 a U_0\left(\mu_n\frac{R_1}{R}\right)\times$$

$$\times\int_0^\tau \psi_1(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\int_0^\tau p(\tau)\,d\tau\right]d\tau+\int_0^\tau \overline{Q_n}(\tau)\exp\left[\mu_n^2\frac{a\tau}{R^2}-\right.$$

$$\left.\left.\left.\left.-\int_0^\tau p(\tau)\,d\tau\right]d\tau\right]\right]U_0\left(\mu_n\frac{\xi}{R}\right)\exp\left[-\mu_n^2\frac{a\tau}{R^2}\right]\right\}.$$

(3-39)

Use of G. A. Greenberg transforms for solution of two-dimensional and three-dimensional problems.

The solution of two-dimensional and three-dimensional problems of heat conductivity can be reduced to repeated integral transforms, i.e., to successive application to the differential equation and edge conditions of the problem of integral transforms with respect to each of the spatial coordinates and inversion with respect to the corresponding formulas.

Let us assume we must solve the following two-dimensional problem:

$$\frac{\partial T}{\partial\tau}=a\left[\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial T}{\partial\xi}\right)+\frac{\partial^2 T}{\partial\zeta^2}\right]+p(\tau)T+Q(\xi,\zeta,\tau)$$

$$(R_1<\xi<R_2,\ 0<\zeta<L,\ \tau>0,\ i=0\vee 1);\qquad (3\text{-}40)$$

$$T(\xi,\zeta,0)=f(\xi,\zeta)\ (R_1\leqslant\xi\leqslant R_2,\ 0\leqslant\zeta\leqslant L);\qquad (3\text{-}41)$$

$$\frac{\partial T(R_j,\zeta,\tau)}{\partial\xi}=(-1)^j h_j[\psi_j(\zeta,\tau)-T(R_j,\zeta,\tau)]\ (j=1,2);$$

(3-42)

$$\frac{\partial T(\xi,(k-3)L,\tau)}{\partial\zeta}=(-1)^k h_k[\psi_k(\xi,\tau)-T(\xi,(k-3)L,\tau)]\ (k=3,4).$$

The first transform is performed with respect to variable ξ. The corresponding Shturm-Liouville problem with respect to this variable is:

$$\frac{d}{d\xi}\left[\xi^i\frac{dU_0\left(\mu_n\frac{\xi}{R}\right)}{d\xi}\right]+\frac{\mu_n^2}{R^2}\xi^i U_0\left(\mu_n\frac{\xi}{R}\right)=0$$

$$(R_1<\xi<R_2,\ i=0\vee 1);\qquad (3\text{-}43)$$

$$\frac{dU_0\left(\mu_n \frac{R_j}{R}\right)}{d\xi}+(-1)^j h_j U_0\left(\mu_n \frac{R_j}{R}\right)=0 \; (j=1,\, 2). \qquad (3\text{-}44)$$

Consequently, the integral transform with respect to variable ξ must be defined by the formulas:

$$\overline{T}_n(\zeta,\, \tau)=\int_{R_1}^{R_2} \xi^i T(\xi,\, \zeta, \tau)\, U_0\left(\mu_n \frac{\xi}{R}\right) d\xi; \qquad (3\text{-}45)$$

$$T(\xi,\, \zeta,\, \tau)=\sum_{n=1}^{\infty} \frac{\overline{T}_n(\zeta, \tau)}{\|U_0\|^2}\, U_0\left(\mu_n \frac{\xi}{R}\right), \qquad (3\text{-}46)$$

where $U_0(\mu_n \frac{\xi}{R})$ is the Eigenfunction of problem (3-43)-(3-44); μ_n is the root of the characteristic equation following from this problem;

$\|U_0\|^2=\int_{R_1}^{R_2} \xi^i U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$ is the square of the norm of the corresponding function $U_0(\mu_n(\xi/R))$.

Differential equation (3-40), initial condition (3-41) and the second boundary condition from (3-42) are multiplied by $\xi^i U_0(\mu_n(\xi/R))$ and integrated within limits from 0 to R.

Integral I is:

$$I=\int_{R_1}^{R_2} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi=\xi^i \frac{\partial T}{\partial \xi} U_0\left(\mu_n \frac{\xi}{R}\right)\Big|_{R_1}^{R_2}-$$

$$-\xi^i T \frac{d}{d\xi} U_0\left(\mu_n \frac{\xi}{R}\right)\Big|_{R_1}^{R_2}+\int_{R_1}^{R_2} T \frac{d}{d\xi}\left[\xi^i \frac{dU_0\left(\mu_n \frac{\xi}{R}\right)}{d\xi}\right] d\xi=$$

$$=R_2^i h_2 a \psi_2(\tau)\, U_0\left(\mu_n \frac{R_2}{R}\right)+R_1^i h_1 a \psi_1(\zeta,\, \tau)\, U_0\left(\mu_n \frac{R_1}{R}\right)-\frac{\mu_n^2}{R^2} \overline{T}_n.$$

Here we keep in mind the first boundary condition (3-42), boundary condition (3-44) as well as equation (3-43). From this for the transform $\overline{T}_n(\zeta,\, \tau)$ we produce the differential equation

$$\frac{\partial \overline{T}_n}{\partial \tau} = a\frac{\partial^2 \overline{T}_n}{\partial \zeta^2} - \frac{a\mu_n^2}{R^2}\overline{T}_n + p(\tau)\overline{T}_n + \overline{Q}_n(\zeta, \tau) +$$

$$+ R_2^i h_2 a\psi_2(\zeta, \tau) U_0\left(\mu_n \frac{R_2}{R}\right) + R_1^i h_1 a\psi_1(\zeta, \tau) U_0\left(\mu_n \frac{R_1}{R}\right)$$

$$(0 < \zeta < L, \ \tau > 0, \ i = 0 \vee 1); \tag{3-47}$$

the initial condition

$$\overline{T}_n(\zeta, 0) = \overline{f}_n(\zeta) \ (0 \leq \zeta \leq L); \tag{3-48}$$

the boundary condition

$$\frac{\partial \overline{T}_n((k-3)L, \tau)}{\partial \zeta} = (-1)^k h_k [\overline{\psi}_{k(n)}(\tau) - \overline{T}_n((k-3)L, \tau)]. \tag{3-49}$$

The Shturm-Liouville problem with respect to variable ζ is:

$$\frac{d^2 V_0\left(\nu_m \frac{\zeta}{L}\right)}{d\zeta^2} + \frac{\nu_m^2}{L^2} V_0\left(\nu_m \frac{\zeta}{L}\right) = 0 \ (0 < \zeta < L);$$

$$\frac{dV_0((k-3)\nu_m)}{d\zeta} + (-1)^k h_k V_0((k-3)\nu_m) = 0 \ (k = 3, 4).$$

Therefore, the integral transform with respect to variable ζ is defined by the formulas

$$\widetilde{T}_{nm}(\tau) = \int_0^L \overline{T}_n(\zeta, \tau) V_0\left(\nu_m \frac{\zeta}{L}\right) d\zeta; \tag{3-50}$$

$$\overline{T}_n(\zeta, \tau) = \sum_{m=1}^{\infty} \frac{\widetilde{T}_{nm}(\tau)}{\|V_0\|^2} V_0\left(\nu_m \frac{\zeta}{L}\right), \tag{3-51}$$

where $V_0(\nu_m \frac{\zeta}{L})$ is the Eigenfunction of the problem; ν_m is the root of the characteristic equation;

$$\|V_0\|^2=\int_0^L V_0^2\left(\nu_m \frac{\zeta}{L}\right) d\zeta.$$

Using the standard plan, equation (3-47) and the initial condition (3-48), we can perform a transform with respect to variable ζ. We find:

$$\frac{d\tilde{\bar{T}}_{nm}}{d\tau}=-\left(\frac{a\mu_n^2}{R^2}+\frac{a\nu_m^2}{L^2}-p(\tau)\right)\tilde{\bar{T}}_{nm}+\tilde{\bar{Q}}_{nm}(\tau)+$$
$$+R_2^i h_2 a\tilde{\psi}_{2(m)}(\tau)\,U_0\left(\mu_n\frac{R_2}{R}\right)+R_1^i h_1 a\tilde{\psi}_{1(m)}(\tau)\,U_0\left(\mu_n\frac{R_1}{R}\right)+$$
$$+h_4 a\bar{\psi}_{4(n)}(\tau)\,V_0(\nu_m)+h_3 a\bar{\psi}_{3(n)}(\tau)\,V_0(0);$$
$$\tilde{\bar{T}}_{nm}(0)=\tilde{\bar{f}}_{nm}.$$

Integrating this ordinary differential equation, we produce

$$\tilde{\bar{T}}_{nm}(\tau)=\tilde{\bar{f}}_{nm}\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]+$$
$$+R_2^i h_2 a U_0\left(\mu_n\frac{R_2}{R}\right)\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\times$$
$$\times\int_0^\tau \tilde{\psi}_{2(m)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$
$$+R_1^i h_1 a U_0\left(\mu_n\frac{R_1}{R}\right)\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\times$$
$$\times\int_0^\tau \tilde{\psi}_{1(m)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$
$$+h_4 a V_0(\nu_m)\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\times$$
$$\times\int_0^\tau \bar{\psi}_{4(n)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$
$$+h_3 a V_0(0)\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\times$$
$$\times\int_0^\tau \bar{\psi}_{3(n)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\int_0^\tau \tilde{\tilde{Q}}_{nm}\times$$

$$\times\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)\tilde{a\tau}-\int_0^\tau p(\tau)\,d\tau\right]d\tau.$$

Successive application of the inverse transforms (3-46) and (3-51) leads us to the solution of the problem:

$$T(\xi,\zeta,\tau)=\exp\left[\int_0^\tau p(\tau)\,d\tau\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{\|U_0\|^2\|V_0\|^2}\left[\left[\tilde{\tilde{f}}_{nm}+\right.\right.\right.$$

$$+R_2^i h_2 aU_0\left(\mu_n\frac{R_2}{R}\right)\int_0^\tau\tilde{\psi}_{2(m)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+R_1^i h_1 aU_0\left(\mu_n\frac{R_1}{R}\right)\int_0^\tau\tilde{\psi}_{1(m)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+h_4 aV_0(\nu_m)\int_0^\tau\overline{\psi}_{4(n)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+h_3 aV_0(0)\int_0^\tau\overline{\psi}_{3(n)}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau+$$

$$+\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau+\int_0^\tau p(\tau)\,d\tau\right]\int_0^\tau\tilde{\tilde{Q}}_{nm}(\tau)\exp\left[\left(\frac{\mu_n^2}{R^2}+\right.\right.$$

$$\left.\left.\left.\left.+\frac{\nu_m^2}{L^2}\right)a\tau-\int_0^\tau p(\tau)\,d\tau\right]d\tau\right]\right]U_0\left(\mu_n\frac{\xi}{R}\right)V_0\left(\nu_m\frac{\zeta}{L}\right)\times$$

$$\times\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{\nu_m^2}{L^2}\right)a\tau\right], \qquad (3\text{-}52)$$

where

$$\tilde{\tilde{f}}_{nm}=\int_{R_1}^{R_2}\int_0^L\xi^i f(\xi,\zeta)\,U_0\left(\mu_n\frac{\xi}{R}\right)V_0\left(\nu_m\frac{\zeta}{L}\right)d\xi\,d\zeta;$$

$$\tilde{\tilde{Q}}_{nm}=\int_{R_1}^{R_2}\int_0^L\xi^i Q(\xi,\zeta,\tau)\,U_0\left(\mu_n\frac{\xi}{R}\right)V_0\left(\nu_m\frac{\zeta}{L}\right)d\xi\,d\zeta;$$

$$\tilde{\psi}_{j(m)}=\int_0^L\psi_j(\zeta,\tau)\,V_0\left(\nu_m\frac{\zeta}{L}\right)d\zeta;$$

$$\overline{\psi}_{k(n)}=\int_{R_1}^{R_2}\xi^i\psi_k(\xi,\tau)\,U_0\left(\mu_n\frac{\xi}{R}\right)d\xi.$$

Example. Establish the temperature field of a concrete wall, one surface of which (x = 0) is maintained at constant temperature T_1, whereas the other (x = R) is cooled in a medium with a temperature of 0. The initial temperature of the wall T_0 = const. In the wall, due to hydration of cement, heat is liberated, the intensity of heat liberation depending exponentially on time.

The mathematical formulation of the problem is as follows:

differential equation

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} + \frac{1}{c\gamma} q_0 e^{-m\tau} \quad (0 < x < R, \tau > 0);$$

initial condition

$$T(x, 0) = T_0 \quad (0 \leqslant x \leqslant R);$$

boundary conditions

$$T(0, \tau) = T_1; \quad \frac{\partial T(R, \tau)}{\partial x} = -hT(R, \tau).$$

The corresponding Shturm-Liouville problem is:

$$\frac{d^2 U_0\left(\mu_n \frac{x}{R}\right)}{dx^2} + \frac{\mu_n^2}{R^2} U_0\left(\mu_n \frac{x}{R}\right) = 0;$$

$$U_0(0) = 0; \quad \frac{dU_0(\mu_n)}{dx} = -hU_0(\mu_n).$$

From this, the Eigenfunction of the problem is:

$$U_0\left(\mu_n \frac{x}{R}\right) = \sin \mu_n \frac{x}{R},$$

where μ_n is the root of the characteristic equation

$$\tan \mu_n = -\frac{\mu_n}{\mathrm{Bi}}; \quad \mathrm{Bi} = hR.$$

Consequently, the integral transform in the problem here under study is defined by the formulas

$$\bar{T}_n(\tau) = \int_0^R T(x, \tau) \sin \mu_n \frac{x}{R}\, dx; \tag{3-53}$$

$$T(x, \tau) = \sum_{n=1}^{\infty} \frac{\bar{T}_n(\tau)}{\| U_0 \|^2} \sin \mu_n \frac{x}{R}, \tag{3-54}$$

where

$$\| U_0 \|^2 = \int_0^R \sin^2 \mu_n \frac{x}{R}\, dx = \frac{R}{2} \frac{\mathrm{Bi}^2 + \mathrm{Bi} + \mu_n^2}{\mathrm{Bi}^2 + \mu_n^2}.$$

Using integral transform (3-53), we produce for the transformation of $\bar{T}_n(\tau)$ the differential equation

$$\frac{d\bar{T}_n}{d\tau} = -\frac{a\mu_n^2}{R^2} \bar{T}_n + \frac{a\mu_n}{R} T_1 + \frac{1}{c\gamma} q_0 e^{-m\tau} N_n$$

and the initial condition

$$\bar{T}_n(0) = T_0 N_n,$$

where

$$N_n = \int_0^R \sin \mu_n \frac{x}{R}\, dx = \frac{R}{\mu_n} \left[1 + \frac{(-1)^{n+1} \mathrm{Bi}}{\sqrt{\mu_n^2 + \mathrm{Bi}^2}} \right].$$

The solution of this equation is

$$\bar{T}_n(\tau) = T_0 N_n \exp\left[-\mu_n^2 \frac{a\tau}{R^2}\right] + \frac{a\mu_n}{R} T_1 \exp\left[-\mu_n^2 \frac{a\tau}{R^2}\right] \times$$
$$\times \int_0^\tau \exp\left[\mu_n^2 \frac{a\tau}{R^2}\right] d\tau + \frac{q_0}{c\gamma} N_n \exp\left[-\mu_n^2 \frac{a\tau}{R^2}\right] \int_0^\tau \exp\left[\left(\frac{\mu_n^2 a}{R^2} - m\right) \tau\right] d\tau. \tag{3-55}$$

From this, based on the inversion formula (3-54), we establish the general solution of the problem

$$T(x,\tau)=\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\left[T_0 N_n-\frac{RT_1}{\mu_n}+\frac{q_0R^2}{\lambda}\frac{N_n}{\mu_n^2-m^{*2}}\right]\sin\mu_n\frac{x}{R}\times$$
$$\times\exp[-\mu_n^2\,\mathrm{Fo}]+T_1\sum_{n=1}^{\infty}\frac{R\sin\mu_n\frac{x}{R}}{\mu_n\|U_0\|^2}+$$
$$+\frac{q_0R^2}{\lambda}\sum_{n=1}^{\infty}\frac{A_n}{\mu_n^2-m^{*2}}\sin\mu_n\frac{x}{R}, \qquad (3\text{-}56)$$

where in addition to the previous symbols

$$A_n=\frac{N_n}{\|U_0\|^2};\quad m^{*2}=\frac{mR^2}{a};\quad \mathrm{Fo}=\frac{a\tau}{R^2}.$$

The second series in expression (3-56) converges slowly (order of convergence not over $1/\mu_n$). However, this series, like, parenthetically, the series which follows it, can be summed.

In order to find the sum of these series, we must analyze two supplementary problems.

Problem 1.

$$\frac{d^2F}{dx^2}=0;\; F(0)=1;\; \frac{dF(R)}{dx}=-hF(R). \qquad (3\text{-}57)$$

Applying the integral transform (3-53) to problem (3-57), we produce

$$-\frac{\mu_n^2}{R^2}\bar{F}_n+\frac{\mu_n}{R}=0$$

or

$$\bar{F}_n=\frac{R}{\mu_n}.$$

From the inversion formula (3-54) it follows that

$$F=\sum_{n=1}^{\infty}\frac{R}{\mu_n\|U_0\|^2}\sin\mu_n\frac{x}{R}. \tag{3-58}$$

On the other hand, problem (3-57) is solved in closed form

$$F=1-\frac{\mathrm{Bi}}{1+\mathrm{Bi}}\frac{x}{R}. \tag{3-59}$$

Due to the uniqueness of the solution, expressions (3-58) and (3-59) are identical, i.e.

$$\sum_{n=1}^{\infty}\frac{R}{\mu_n\|U_0\|^2}\sin\mu_n\frac{x}{R}=1-\frac{\mathrm{Bi}}{1+\mathrm{Bi}}\frac{x}{R}. \tag{3-60}$$

Problem 2.

$$\frac{d^2F^2}{dx^2}+\frac{m}{a}F=-1;$$
$$F(0)=0;\quad \frac{dF(R)}{dx}=-hF(R). \tag{3-61}$$

The G. A. Greenberg transform yields:

$$-\frac{\mu_n^2}{R^2}\bar{F}_n+\frac{m}{a}\bar{F}_n=-N_n$$

or

$$\bar{F}_n=\frac{R^2N_n}{\mu_n^2-m^{*2}},$$

where $m^{*2}=\dfrac{mR^2}{a}$.

In accordance with the inversion formula (3-54)

$$F = R^2 \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 - m^{*2}} \sin \mu_n \frac{x}{R}. \tag{3-62}$$

Comparing the solution of problem (3-61) in closed form with the solution of (3-62), we find:

$$\sum_{n=1}^{\infty} \frac{A_n U_0 \left(\mu_n \frac{x}{R}\right)}{\mu_n^2 - m^{*2}} = \frac{1}{m^{*2}} \left[\cos m^* \frac{x}{R} + \frac{m^* \sin m^* - \mathrm{Bi} \cos m^* + \mathrm{Bi}}{m^* \cos m^* + \mathrm{Bi} \sin m^*} \times \right.$$

$$\left. \times \sin m^* \frac{x}{R} - 1 \right]. \tag{3-63}$$

Formulas (3-60) and (3-63) are the desired summation formulas

Improvement of the Convergence of Series

In solutions of the problem of heat conductivity, when the finite integral G. A. Greenberg transforms are directly used, weakly converging series are frequently produced. We encountered examples of such series in the preceding section.

We suggest below methods for improving the convergence of such series.

For simplicity of presentation, we will study separately problems with heterogeneous differential equations and homogeneous boundary conditions, and problems with homogeneous differential equations and heterogeneous boundary conditions.

The boundary conditions are formulated in a general form. In connection with this, the finite integral transforms used are defined by the expressions:

$$\bar{u}_n = \int_{R_1}^{R_2} \xi^i u U_0 \left(\mu_n \frac{\xi}{R}\right) d\xi \, (i = 0 \vee 1);$$

$$u = \sum_{n=1}^{\infty} \frac{\bar{u}_n}{\|U_0\|^2} U_0 \left(\mu_n \frac{\xi}{R}\right),$$

where $U_0(\mu_n(\xi/R))$ is the Eigenfunction of the problem; μ_n is the root of the characteristic equations; R is the characteristic dimension of the body;

$$\|U_0\|^2=\int_{R_1}^{R_2}\xi^i U_0^2\left(\mu_n\frac{\xi}{R}\right)d\xi.$$

Problems with heterogeneous differential equations and homogeneous boundary conditions such as

$$\frac{\partial T}{\partial \tau}=a\nabla^2 T+\frac{1}{c\gamma}q(\xi,\tau)\ (R_1<\xi<R_2,\tau>0); \tag{3-64}$$

$$T(\xi,0)=0\ (R_1\leqslant\xi\leqslant R_2); \tag{3-65}$$

$$\alpha_j\frac{\partial T(R_j,\tau)}{\partial\xi}+(-1)^j\beta_j T(R_j,\tau)=0\ (j=1,2)^*. \tag{3-66}$$

Here

$$\nabla^2=\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial}{\partial\xi}\right)\ (i=0\vee 1).$$

Problem 1. If

$$q(\xi,\tau)=\rho(\xi),$$

then differential equation (3-64) becomes

$$\frac{\partial T}{\partial \tau}=a\nabla^2 T+\frac{1}{c\gamma}\rho(\xi)\ (R_1<\xi<R_2,\tau>0). \tag{3-67}$$

---

[1]The form of inscription of (3-66) allows various combinations of boundary conditions of first, second and third kind. However, in § 3-4, boundary conditions of the second kind at both ends of the interval $[R_1, R_2]$ were not specially studied. This case corresponds to complete thermal insulation of the body, as a result of which we need but add the following term to the temperature functions produced below

$$T_{\text{sop}}=\frac{\int_{R_1}^{R_2}\int_0^{\tau}\xi^i q(\xi,\tau)\,d\xi\,d\tau}{c\gamma\int_{R_1}^{R_2}\xi^i d\xi}.$$

Let us assume:

$$T = \theta(\xi, \tau) + v_0(\xi), \tag{3-68}$$

where $v_0(\xi)$ is an as yet arbitrary, twice differentiable function.

We have:

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta + a\nabla^2 v_0 + \frac{1}{c\gamma}\rho(\xi).$$

It is natural to require that

$$\nabla^2 v_0 = -\frac{1}{\lambda}\rho(\xi) \quad (R_1 < \xi < R_2). \tag{3-69}$$

Then

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta \quad (R_1 < \xi < R_2, \tau > 0);$$

$$\theta(\xi, 0) = -v_0 \quad (R_1 \leqslant \xi \leqslant R_2). \tag{3-70}$$

The boundary conditions for functions v_0 and θ are homogeneous, similar to (3-66).

Integral transformation of problems (3-69) and (3-70) yields

$$-\frac{\mu_n^2}{R^2}\bar{v}_{0(n)} = -\frac{\bar{\rho}_n}{\lambda}$$

and

$$\frac{d\bar{\theta}_n}{d\tau} = -\frac{a\mu_n^2}{R^2}\bar{\theta}_n.$$

From this, the mappings $\bar{v}_0(n)$ and $\bar{\theta}_n$ are:

$$\bar{v}_{0(n)} = \frac{R^2}{\lambda} \frac{\bar{p}_n}{\mu_n^2};$$

$$\bar{\theta}_n = -\frac{R^2}{\lambda} \frac{\bar{p}_n}{\mu_n^2} \exp\left[-\mu_n^2 \frac{a\tau}{R^2}\right],$$

and their originals v_0 and θ are expressed by the formulas

$$v_0 = \frac{R^2}{\lambda} \sum_{n=1}^{\infty} \frac{\bar{p}_n}{\mu_n^2 \| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right); \tag{3-71}$$

$$\theta = -\frac{R^2}{\lambda} \sum_{n=1}^{\infty} \frac{\bar{p}_n}{\mu_n^2 \| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \exp\left[-\mu_n^2 \frac{a\tau}{R^2}\right]. \tag{3-72}$$

The sum

$$T = \theta + v_0$$

is the solution of problem (3-64)-(3-66).

Series (3-71) converges as $1/\mu_n^2$. For its summation, we should use the solution of equation (3-69) with boundary conditions such as (3-66) in closed form and compare it to the solution in the form of (3-71). Due to uniqueness, these solutions are identical.

Thus, the sum of the series

$$R^2 \sum_{n=1}^{\infty} \frac{\bar{p}_n}{\mu_n^2 \| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \tag{3-73}$$

is the solution (in closed form) of the problem

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw}{d\xi}\right) = -p(\xi) \quad (i = 0 \vee 1);$$

$$\alpha_j \frac{dw(R_j)}{d\xi} + (-1)^j \beta_j w(R_j) = 0 \quad (j = 1, 2). \tag{3-74}$$

Let us assume

$$\rho(\xi) = 1.$$

Then

$$\bar{\rho}_n = \int_{R_1}^{R_2} \xi^i \rho(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \int_{R_1}^{R_2} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = N_n.$$

Consequently, the summation formula of the series, frequently encountered in practice,

$$R^2 \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right); \; A_n = \frac{N_n}{\| U_0 \|^2}$$

is produced as a result of integration of the equation

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw}{d\xi}\right) = -1$$

with the boundary conditions of problem (3-74).

For example, with boundary conditions of the third kind:

for a wall $(0 \leq x \leq R)$

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{x}{R}\right) = \frac{1}{2}\left[C\left(1 + \mathrm{Bi}_1 \frac{x}{R}\right) - \frac{x^2}{R^2}\right],$$

where

$$C = \frac{2 + \mathrm{Bi}_2}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2}; \; \mathrm{Bi}_1 = h_1 R; \; \mathrm{Bi}_2 = h_2 R;$$

for a solid cylinder ($0 \le r \le R$)

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{r}{R}\right) = \frac{1}{4}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}}\right); \ \mathrm{Bi} = hR;$$

for a hollow cylinder ($R_1 \le r \le R_2$)

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{r}{R}\right) = \frac{1}{4}\left(1 + D \ln \frac{r}{R_1} - \frac{r^2}{R_1^2} + \frac{D-2}{\mathrm{Bi}_1}\right),$$

where

$$D = \frac{2k\,\mathrm{Bi}_1 + 2\,\mathrm{Bi}_2 + (k^2 - 1)\,\mathrm{Bi}_1\,\mathrm{Bi}_2}{k\,\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1\mathrm{Bi}_2}; \ \mathrm{Bi}_1 = h_1 R_1; \ \mathrm{Bi}_2 = h_2 R_1; \ k = \frac{R_2}{R_1}.$$

Problem 2. Suppose $q(\xi, \tau) = \rho(\xi) \sum_{s=0}^{m} q_s(\xi)\tau^s$ (m power polynomial relative to τ).

We assume:

$$T = \theta(\xi, \tau) + \sum_{s=0}^{m} v_s(\xi)\tau^s.$$

We have:

$$\frac{\partial\theta}{\partial\tau} + \sum_{s=0}^{m} s v_s \tau^{s-1} = a\nabla^2\theta + a\sum_{s=0}^{m} \tau^s \nabla^2 v_s + \frac{1}{c\gamma}\,\rho(\xi) \sum_{s=0}^{m} q_s \tau^s.$$

We accept:

$$\nabla^2 v_m = -\frac{q_m}{\lambda}\rho(\xi);$$

$$\nabla^2 v_{m-1} = -\frac{q_{m-1}}{\lambda}\rho(\xi) + \frac{m}{a} v_m.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\nabla^2 v_0 = -\frac{q_0}{\lambda}\rho(\xi) + \frac{1}{a} v_1.$$

Then

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta; \quad \theta(\xi, 0) = -v_0(\xi).$$

The boundary conditions for the equations presented are homogeneous. After integral transformation of this system of equations, its solution for the mappings, and inversion of the mappings, we produce:

$$v_m = \frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n q_m}{\mu_n^2 \| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right);$$

$$v_{m-1} = \frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n}{\| U_0 \|^2}\left[\frac{q_{m-1}}{\mu_n^2} - \frac{mq_m R^2}{a\mu_n^4}\right] U_0\left(\mu_n \frac{\xi}{R}\right);$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_0 = \frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n}{\| U_0 \|^2}\left[\frac{q_0}{\mu_n^2} - \frac{q_1 R^2}{a\mu_n^4} + \cdots + (-1)^m \frac{m! q_m R^{2m}}{a^m \mu_n^{2(m+1)}}\right] \times$$
$$\times U_0\left(\mu_n \frac{\xi}{R}\right)$$

and

$$\theta = -\frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n}{\| U_0 \|^2}\left[\frac{q_0}{\mu_n^2} - \frac{q_1 R^2}{a\mu_n^4} + \cdots + (-1)^m \frac{m\, q_m R^{2m}}{a^m \mu_n^{2(m+1)}}\right] \times$$
$$\times U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Thus for summation of the series

$$(-1)^s R^{2(s+1)} \sum_{n=1}^{\infty} \frac{\bar{p}_n}{\mu_n^{2(s+1)} \| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \quad (s=1, 2...) \tag{3-75}$$

we must integrate the system

$$\begin{aligned}
&\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw_s}{d\xi}\right) = -\rho(\xi);\\
&\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw_{s-1}}{d\xi}\right) = w_s;\\
&\cdots\cdots\cdots\cdots\\
&\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw_0}{d\xi}\right) = w_1
\end{aligned} \tag{3-76}$$

with homogeneous boundary conditions. The solution for function $w_0(\xi)$ will be the sum of the series (3-75).

If, for example,

$$q(\xi, \tau) = q_0 + q_1 \tau,$$

the solution of the corresponding heat conductivity problem will be:

$$T = \theta + v_0 + v_1 \tau,$$

where

$$v_0 = \frac{q_0 R^2}{\lambda} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right) - \frac{q_1 R^4}{a\lambda} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^4} U_0\left(\mu_n \frac{\xi}{R}\right);$$

$$v_1 = \frac{q_1 R^2}{\lambda} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right);$$

$$\theta = -\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2}\left[\frac{q_0 R^2}{\lambda} - \frac{q_1 R^4}{a\lambda} \frac{1}{\mu_n^2}\right] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

The formula for summation of the series

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right)$$

was presented earlier.

The sum of the series

$$-R^4 \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^4} U_0\left(\mu_n \frac{\xi}{R}\right)$$

is the solution w_0 of the equation system

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw_1}{d\xi}\right) = -1;$$
$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw_0}{d\xi}\right) = w_1$$

with homogeneous boundary conditions.

For example, for a wall with boundary conditions of the second (x = 0) and third (x = R) kinds, the summation formula has the final form:

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^4} U_0\left(\mu_n \frac{x}{R}\right) = \frac{1}{24}\left[\frac{x^4}{R^4} - \frac{6(\mathrm{Bi}+2)}{\mathrm{Bi}} \frac{x^2}{R^2} + \right.$$
$$\left. + \frac{1}{\mathrm{Bi}}\left(5\,\mathrm{Bi} + 20 + \frac{24}{\mathrm{Bi}}\right)\right].$$

Problem 3. The heat liberation intensity function is

$$q(\xi, \tau) = \rho(\xi)\, e^{\mp m\tau}.$$

We assume:

$$T = \theta(\xi, \tau) + v_0(\xi)\, e^{\mp m\tau}.$$

We have:

$$\frac{\partial \theta}{\partial \tau} \mp m v_0 e^{\mp m\tau} = a\nabla^2\theta + e^{\mp m\tau}\,\nabla^2 v_0 + \frac{1}{c\gamma}\,\rho(\xi)\,e^{\mp m\tau}.$$

We therefore accept:

$$\nabla^2 v_0 \pm \frac{m}{a}\,v_0 = -\frac{\rho(\xi)}{\lambda};$$

$$\frac{\partial \theta}{\partial \tau} = a\nabla^2\theta;$$

$$\theta(\xi, 0) = -v_0.$$

The integral transform of the latter problems yields:

$$\bar{v}_{0(n)} = \frac{R^2}{\lambda}\,\frac{\bar{\rho}_n}{\mu_n^2 \mp m^{*2}};$$

$$\bar{\theta}_n = -\frac{\rho^2}{\lambda}\,\frac{\bar{\rho}_n}{\mu_n^2 \mp m^{*2}}\,e^{-\mu_n^2\frac{a\tau}{R^2}},$$

where

$$m^{*2} = \frac{mR^2}{a}.$$

From this

$$v_0 = \frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n}{(\mu_n^2 \mp m^{*2})\,\|U_0\|^2}\,U_0\left(\mu_n\frac{\xi}{R}\right);$$

$$\theta = -\frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{\bar{\rho}_n}{(\mu_n^2 \mp m^{*2})\,\|U_0\|^2}\,U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}}.$$

Thus, the sum of the series

$$R^2 \sum_{n=1}^{\infty} \frac{\bar{p}_n}{(\mu_n^2 \mp m^{*2}) \| U_0 \|^2} U_0 \left(\mu_n \frac{\xi}{R} \right) \tag{3-77}$$

is the solution of the equation

$$\frac{1}{\xi^i} \frac{d}{d\xi} \left(\xi^i \frac{dw}{d\xi} \right) \pm \frac{m^{*2}}{R^2} w = -\rho(\xi) \tag{3-78}$$

with homogeneous boundary conditions.

Problem 4. The heat liberation intensity function is:

$$q(\xi, \tau) = \rho(\xi) \cos(\omega\tau + \varepsilon),$$

where ω is the frequency; ε is the initial phase.

Let us represent the heat liberation intensity function in complex form[1]

$$q(\xi, \tau) = \mathrm{Re}\, \rho(\xi) e^{i(\omega\tau + \varepsilon)}.$$

We assume:

$$T = \theta(\xi, \tau) + v_0(\xi) e^{i(\omega\tau + \varepsilon)}.$$

We have:

$$\frac{\partial \theta}{\partial \tau} + i\omega v_0 e^{i(\omega\tau + \varepsilon)} = a\nabla^2\theta + ae^{i(\omega\tau + \varepsilon)} \nabla^2 v_0 + \frac{1}{c\gamma} \rho(\xi) e^{i(\omega\tau + \varepsilon)}.$$

We accept:

$$\nabla^2 v_0 - i\frac{\omega}{a} v_0 = -\frac{\rho(\xi)}{\lambda}.$$

[1]Re is the symbol for the real portion of the complex number, i is the imaginary unit.

Then

$$\frac{\partial\theta}{\partial\tau}=a\nabla^2\theta;$$
$$\theta(\xi, 0)=-v_0 e^{i\omega}.$$

From this

$$\bar{v}_{0(n)}=\frac{R^2}{\lambda}\frac{\bar{\rho}_n}{\mu_n^2+i\omega^{*2}};$$
$$\bar{\theta}_n=-\frac{R^2}{\lambda}\frac{\bar{\rho}_n}{\mu_n^2+i\omega^{*2}}e^{-\mu_n^2\frac{a\tau}{R^2}}$$

or

$$\bar{v}_{0(n)}=\frac{R^2}{\lambda}\frac{\mu_n^2\bar{\rho}_n}{\mu_n^4+\omega^{*4}}-i\frac{R^2}{\lambda}\frac{\omega^{*2}\bar{\rho}_n}{\mu_n^4+\omega^{*4}};$$
$$\bar{\theta}_n=-\left[\frac{R^2}{\lambda}\frac{\mu_n^2\bar{\rho}_n}{\mu_n^4+\omega^{*4}}e^{-\mu_n^2\frac{a\tau}{R^2}}-i\frac{R^2}{\lambda}\frac{\omega^{*2}\bar{\rho}_n}{\mu_n^4+\omega^{*4}}e^{-\mu_n^2\frac{a\tau}{R^2}}\right]e^{i\omega}.$$

Here

$$\omega^{*2}=\frac{\omega R^2}{a}.$$

Based on the inversion formula, we produce:

$$v_0=\frac{R^2}{\lambda}w_1(\xi)-i\frac{R^2}{\lambda}w_2(\xi);$$
$$\theta=-\left[\frac{R^2}{\lambda}w_3(\xi,\tau)-i\frac{R^2}{\lambda}w_4(\xi,\tau)\right]e^{i\omega},$$

where

$$w_1(\xi)=\sum_{n=1}^{\infty}\frac{\mu_n^2\,\bar{p}_n}{(\mu_n^4+\omega^{*4})\,\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right);$$

$$w_2(\xi)=\sum_{n=1}^{\infty}\frac{\omega^{*2}\bar{p}_n}{(\mu_n^4+\omega^{*4})\,\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right);$$

$$w_3(\xi,\tau)=\sum_{n=1}^{\infty}\frac{\mu_n^2\,\bar{p}_n}{(\mu_n^4+\omega^{*4})\,\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}};$$

$$w_4(\xi,\tau)=\sum_{n=1}^{\infty}\frac{\omega^{*2}\bar{p}_n}{(\mu_n^4+\omega^{*4})\,\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}}.$$

We know that

$$e^{i\alpha}=\cos\alpha+i\sin\alpha.$$

Therefore, the solution of the heat conductivity problem for a body with internal heat sources, the intensity of which changes in time according to a cosine rule $q(\xi, \tau) = \rho(\xi) \cos(\omega\tau + \varepsilon)$, is:

$$\operatorname{Re}T=\frac{R^2}{\lambda}[w_1(\xi)\cos(\omega\tau+\varepsilon)+w_2(\xi)\sin(\omega\tau+\varepsilon)-$$
$$-w_3(\xi,\tau)\cos\varepsilon-w_4(\xi,\tau)\sin\varepsilon]$$

or

$$\operatorname{Re}T=\frac{R^2}{\lambda}\sqrt{w_1^2(\xi)+w_2(\xi)}\cos(\omega\tau+\varepsilon-\varphi)-$$
$$-\frac{R^2}{\lambda}[w_3(\xi,\tau)\cos\varepsilon+w_4(\xi,\tau)\sin\varepsilon],$$

where

$$\phi = \arctan\frac{w_2(\xi)}{w_1(\xi)}.$$

It follows from all of the above that the sum of the series

$$R^2 \sum_{n=1}^{\infty} \frac{\mu_n^2 \bar{\rho}_n}{(\mu_n^4 + \omega^4) \| U_0 \|^2} U_0 \left(\mu_n \frac{\xi}{R} \right) \tag{3-79}$$

and

$$R^2 \sum_{n=1}^{\infty} \frac{\omega^2 \bar{\rho}_n}{(\mu_n^4 + \omega^4) \| U_0 \|^2} U_0 \left(\mu_n \frac{\xi}{R} \right) \tag{3-80}$$

are the real and imaginary parts of the solution of the equation

$$\frac{1}{\xi^s} \frac{d}{d\xi} \left(\xi^s \frac{dw}{d\xi} \right) - i \frac{\omega^2}{R^2} w = -\rho(\xi) \quad (s = 0 \vee 1) \tag{3-81}$$

with homogeneous boundary conditions.

Problems with homogeneous differential equations and heterogeneous boundary conditions such as

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T \quad (R_1 < \xi < R_2, \tau > 0); \tag{3-82}$$

$$T(\xi, 0) = 0 \quad (R_1 \leqslant \xi \leqslant R_2); \tag{3-83}$$

$$\alpha_j \frac{\partial T(R_j, \tau)}{\partial \xi} + (-1)^j \beta_j T(R_j, \tau) = \gamma_j g_j(\tau) \quad (j = 1, 2). \tag{3-84}$$

Suppose

$$T = \Phi(\xi, \tau) - \theta(\xi, \tau), \tag{3-85}$$

where $\Phi(\xi, \tau)$ is a permutation function twice differentiable with respect to ξ and once with respect to τ.

Then, differential equation (3-82), initial condition (3-83) and boundary conditions (3-84) become

$$\frac{\partial \Phi}{\partial \tau} - \frac{\partial \theta}{\partial \tau} = a \nabla^2 \Phi - a \nabla^2 \theta \quad (R_1 < \xi < R_2, \tau > 0); \tag{3-86}$$

$$\theta(\xi, 0) = \Phi(\xi, 0) \quad (R_1 \leqslant \xi \leqslant R_2); \tag{3-87}$$

$$\alpha_j \frac{\partial\Phi(R_j,\tau)}{\partial\xi} - \alpha_j \frac{\partial\theta(R_j,\tau)}{\partial\xi} + (-1)^j\beta_j\Phi(R_j,\tau) - -(-1)^j\beta_j\theta(R_j,\tau) = \gamma_j g_j(\tau) \quad (j=1,2). \tag{3-88}$$

We require that

$$\alpha_j \frac{\partial\Phi(R_j,\tau)}{\partial\xi} + (-1)^j\Phi(R_j,\tau) = \gamma_j g_j(\tau). \tag{3-89}$$

Then

$$\alpha_j \frac{\partial\theta(R_j,\tau)}{\partial\xi} + (-1)^j\theta(R_j,\tau) = 0 \quad (j=1,2). \tag{3-90}$$

Tables 3-1 and 3-2 present the general form of the permutation functions $\Phi(\xi, \tau)$ for a wall, hollow cylinder and solid cylinder, satisfying requirement (3-89). Permutation functions $\Phi(\xi, \tau)$ are expressed through the heterogeneous terms of the boundary conditions and the supplementary, twice differentiable F functions $F_a(\xi)$ and $F_b(\xi)$, each of which satisfies specific boundary conditions -- a homogeneous boundary condition at one end of the interval and a unit[1] heterogeneous condition at the other.

The essence of the method of improving convergence of series by means of permutation functions can be explained on the example of solution of a problem with boundary conditions of the third kind.

The permutation function is taken as

$$\Phi(\xi,\tau) = \psi_1 + (\psi_2 - \psi_1)F_a(\xi).$$

Differential equation (3-86) then becomes:

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta - a(\psi_2 - \psi_1)\nabla^2F_a(\xi) + \psi'_1 + (\psi'_2 - \psi'_1)F_a(\xi). \tag{3-91}$$

Here $F_a(\xi)$ is an as yet arbitrary function, twice differentiable with respect to ξ, satisfying the boundary conditions

[1]The specific unit heterogeneous boundary condition is taken to mean that, depending on the type of boundary conditions, the surface temperature ϕ, ambient temperature ψ or heat flux η are assumed equal to unity.

TABLE 3-1. GENERAL FORM OF THE PERMUTATION FUNCTION $\Phi(\xi, \tau)$ FOR A WALL AND A HOLLOW CYLINDER

Boundary Conditions of Problem			General form of Permutation Function $\Phi(\xi, \tau)$	Boundary Conditions for F Function	
Symbol	Coord. of Bnd.	Boundary Conditions		$F_a(\xi)$	$F_{\text{б}}(\xi)$
I — I	$\xi = R_1$ $\xi = R_2$	$T = \varphi_1(\tau)$ $T = \varphi_2(\tau)$	а) $\varphi_1 + (\varphi_2 - \varphi_1) F_a(\xi)$ б) $\varphi_2 + (\varphi_1 - \varphi_2) F_{\text{б}}(\xi)$ в) $\varphi_2 F_a(\xi) + \varphi_1 F_{\text{б}}(\xi)$	$F_a(R_1) = 0$ $F_a'(R_2) = 1$	$F_{\text{б}}(R_1) = 1$ $F_{\text{б}}(R_2) = 0$
II — II	$\xi = R_1$ $\xi = R_2$	$\frac{\partial T}{\partial \xi} = -\frac{1}{\lambda}\eta_1(\tau)$ $\frac{\partial T}{\partial \xi} = \frac{1}{\lambda}\eta_2(\tau)$	$\eta_2 F_a(\xi) + \eta_1 F_{\text{б}}(\xi)$	$\frac{dF_a(R_1)}{d\xi} = 0$ $\frac{dF_a(R_2)}{d\xi} = \frac{1}{\lambda}$	$\frac{dF_{\text{б}}(R_1)}{d\xi} = -\frac{1}{\lambda}$ $\frac{dF_{\text{б}}(R_2)}{d\xi} = 0$
III — III	$\xi = R_1$ $\xi = R_2$	$\frac{\partial T}{\partial \xi} = -h_1[\psi_1(\tau) - T]$ $\frac{\partial T}{\partial \xi} = h_2[\psi_2(\tau) - T]$	а) $\psi_1 + (\psi_2 - \psi_1) F_a(\xi)$ б) $\psi_2 + (\psi_1 - \psi_2) F_{\text{б}}(\xi)$ в) $\psi_2 F_a(\xi) + \psi_1 F_{\text{б}}(\xi)$	$\frac{dF_a(R_1)}{d\xi} = h_1 F_a(R_1)$ $\frac{dF_a(R_2)}{d\xi} = h_2[1 - F_a(R_2)]$	$\frac{dF_{\text{б}}(R_1)}{d\xi} = -h_1[1 - F_{\text{б}}(R_1)]$ $\frac{dF_{\text{б}}(R_2)}{d\xi} = -h_2 F_{\text{б}}(R_2)$
I — II	$\xi = R_1$ $\xi = R_2$	$T = \varphi_1(\tau)$ $\frac{\partial T}{\partial \xi} = \frac{1}{\lambda}\eta_2(\tau)$	а) $\varphi_1 + \eta_2 F_a(\xi)$ в) $\varphi_1 F_{\text{б}}(\xi) + \eta_2 F_a(\xi)$	$F_a(R_1) = 0$ $\frac{dF_a(R_2)}{d\xi} = \frac{1}{\lambda}$	$F_{\text{б}}(R_1) = 1$ $\frac{dF_{\text{б}}(R_2)}{d\xi} = 0$
II — I	$\xi = R_1$ $\xi = R_2$	$\frac{\partial T}{\partial \xi} = -\frac{1}{\lambda}\eta_1(\tau)$ $T = \varphi_2(\tau)$	б) $\varphi_2 + \eta_1 F_{\text{б}}(\xi)$ в) $\varphi_2 F_a(\xi) + \eta_1 F_{\text{б}}(\xi)$	$\frac{dF_a(R_1)}{d\xi} = 0$ $F_a(R_2) = 1$	$\frac{dF_{\text{б}}(R_1)}{d\xi} = -\frac{1}{\lambda}$ $F_{\text{б}}(R_2) = 0$

I—III	$\xi = R_1$ $\xi = R_2$	$T = \varphi_1(\tau)$ $\dfrac{\partial T}{\partial \xi} = h_2[\psi_2(\tau) - T]$	а) $\varphi_1 + (\psi_2 - \varphi_1) F_a(\xi)$ б) $\psi_2 + (\varphi_1 - \psi_2) F_\text{б}(\xi)$ в) $\psi_2 F_a(\xi) + \varphi_1 F_\text{б}(\xi)$	$F_a(R_1) = 0$ $\dfrac{dF_a(R_2)}{d\xi} = h_2[1 - F_a(R_2)]$	$F_\text{б}(R_1) = 1$ $\dfrac{dF_\text{б}(R_2)}{d\xi} = -h_2 F_\text{б}(R_2)$
III—I	$\xi = R_1$ $\xi = R_2$	$\dfrac{\partial T}{\partial \xi} = -h_1[\psi_1(\tau) - T]$ $T = \varphi_2(\tau)$	а) $\psi_1 + (\varphi_2 - \psi_1) F_a(\xi)$ б) $\varphi_2 + (\psi_1 - \varphi_2) F_\text{б}(\xi)$ в) $\varphi_2 F_a(\xi) + \psi_1 F_\text{б}(\xi)$	$\dfrac{dF_a(R_1)}{d\xi} = h_1 F_a(R_1)$ $F_a(R_2) = 1$	$\dfrac{dF_\text{б}(R_1)}{d\xi} = -h_1[1 - F_\text{б}(R_1)]$ $F_\text{б}(R_2) = 0$
II—III	$\xi = R_1$ $\xi = R_2$	$\dfrac{\partial T}{\partial \xi} = -\dfrac{1}{\lambda}\eta_1(\tau)$ $\dfrac{\partial T}{\partial \xi} = h_2[\psi_2(\tau) - T]$	б) $\psi_2 + \eta_1 F_\text{б}(\xi)$ в) $\psi_2 F_a(\xi) + \eta_1 F_\text{б}(\xi)$	$\dfrac{dF_a(R_1)}{d\xi} = 0$ $\dfrac{dF_a(R_2)}{d\xi} = h_2[1 - F_a(R_2)]$	$\dfrac{dF_\text{б}(R_1)}{d\xi} = -\dfrac{1}{\lambda}$ $\dfrac{dF_\text{б}(R_2)}{d\xi} = -h_2 F_\text{б}(R_2)$
III—II	$\xi = R_1$ $\xi = R_2$	$\dfrac{\partial T}{\partial \xi} = -h_1[\psi_1(\tau) - T]$ $\dfrac{\partial T}{\partial \xi} = \dfrac{1}{\lambda}\eta_2(\tau)$	а) $\psi_1 + \eta_2 F_a(\xi)$ в) $\eta_2 F_a(\xi) + \psi_1 F_\text{б}(\xi)$	$\dfrac{dF_a(R_1)}{d\xi} = h_1 F_a(R_1)$ $\dfrac{dF_a(R_2)}{d\xi} = \dfrac{1}{\lambda}$	$\dfrac{dF_\text{б}(R_1)}{d\xi} = -h_1[1 - F_\text{б}(R_1)]$ $\dfrac{dF_\text{б}(R_2)}{d\xi} = 0$

Note. For a wall $\xi = x$; $R_1 = 0$; R_2 is the width, characteristic dimension $R_2 = R$. For a hollow cylinder $\xi = r$; R_1 is the radius of the internal surface; R_2 is the radius of the outer surface, characteristic dimension R_1.

Boundary conditions for the wall and hollow cylinder written with Roman numerals: I (first kind); II (second kind); III (third kind), the first being the boundary condition where $\xi = R_1$, the second -- where $\xi = R_2$.

Thus, the note "boundary condition III-II" means that boundary conditions of the third kind are fixed at the end $\xi = R_1$, while boundary conditions of the second kind are fixed at the end $\xi = R_2$.

TABLE 3-2. GENERAL FORM OF PERMUTATION FUNCTION OF A SOLID CYLINDER

Boundary Conditions of Prob.			Gen. form of Perm. Function $\Phi(r, \tau)$	Boundary Cond. F Func.
Symbol	Coor. of Boun.	Boundary Condition		$F_a(r)$
I	$r=R$	$T=\varphi(\tau)$	б) φ	$F_a(R)=1$
	$r=0$	T finite	в) $\varphi F_a(r)$	$F_a(0)$ finite
II	$r=R$	$\frac{\partial T}{\partial r}=\frac{1}{\lambda}\eta(\tau)$	$\eta F_a(r)$	$\frac{dF_a(R)}{dr}=\frac{1}{\lambda}$
	$r=0$	T finite		$F_a(0)$ finite
III	$r=R$	$\frac{\partial T}{\partial r}=h[\psi(\tau)-T]$	б) ψ	$\frac{dF_a(R)}{dr}=h[1-F_a(R)]$
	$r=0$	T finite	в) $\psi F_a(r)$	$F_a(0)$ finite

$$\frac{dF_a(R_1)}{d\xi}=h_1F_a(R_1);\quad \frac{dF_a(R_2)}{d\xi}=h_2[1-F_a(R_2)]. \tag{3-92}$$

Let us select function $F_a(\xi)$ such that differential equation (3-91) is simplified as much as possible.

Problem 1. Suppose $\psi_j = T_j$ = const (j = 1, 2).

Then

$$\begin{gathered}\frac{\partial\theta}{\partial\tau}=a\nabla^2\theta-a(T_2-T_1)\nabla^2F_a(\xi)\ (R_1<\xi<R_2,\ \tau>0);\\ \theta(\xi,0)=T_1+(T_2-T_1)F_a(\xi)\ (R_1\leqslant\xi\leqslant R_2);\\ \frac{\partial\theta(R_j,\tau)}{\partial\xi}+(-1)^jh_j\theta(R_j,\tau)=0\ (j=1,2).\end{gathered} \tag{3-93}$$

We require that

$$\nabla^2F_a=\frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i\frac{dF_a}{d\xi}\right)=0. \tag{3-94}$$

In this case, the differential equation in problem (3-93) becomes homogeneous. It follows from this that the permutation function has fully separated the summable portion of the solution.

As we can easily see, where $\psi_j = T_j$ = const (j = 1, 2), function θ is equal to

$$\theta = \sum_{n=1}^{\infty} \frac{T_1 N_n + (T_2 - T_1)\bar{F}_a}{\| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}},$$

where

$$\bar{F}_a = \int_{R_1}^{R_2} \xi^i F_a(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi,$$

the remaining symbols are the same as earlier.

In passing, we note the simple method of calculation $\bar{F}_a$. Applying the integral transform to equation (3-94), considering boundary conditions (3-92) we find:

$$-\frac{\mu_n^2}{R^2} \bar{F}_a + R_2^i h_2 U_0\left(\mu_n \frac{R_2}{R}\right) = 0,$$

from which

$$\bar{F}_a = \frac{1}{\mu_n^2} R R_2^i \, \mathrm{Bi}_2 \, U_0\left(\mu_n \frac{R_2}{R}\right), \quad \mathrm{Bi}_2 = h_2 R.$$

Permutation function $\Phi(\xi, \tau)$ can be accepted in another form

$$\Phi(\xi, \tau) = \psi_2 + (\psi_1 - \psi_2) F_b(\xi),$$

where $F_b(\xi)$ is a function, twice differentiable with respect to ξ, satisfying the boundary conditions:

$$\frac{dF_б(R_1)}{d\xi} = -h_1[1 - F_б(R_1)]; \quad \frac{dF_б(R_2)}{d\xi} = -h_2 F_б(R_2). \tag{3-95}$$

Assuming

$$\nabla^2 F_б(\xi) = 0,$$

we produce for determination of function $\theta(\xi, \tau)$ the differential equation

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta + \psi'_2 + (\psi'_1 - \psi'_2) F_б(\xi);$$

the initial condition

$$\theta(\xi, 0) = \psi_2(0) + [\psi_1(0) - \psi_2(0)] F_б(\xi)$$

and homogeneous boundary conditions.

The solution of the last problem where $\psi_j = T_j = \text{const}$ $(j = 1, 2)$ is:

$$\theta = \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} [T_2 N_n + (T_1 - T_2) F_б] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Recalling that function $F_b(\xi)$ is an integral of the differential equation

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dF_б}{d\xi}\right) = 0$$

with boundary conditions (3-95), we easily find:

$$F_б = \frac{1}{\mu_n^2} R R_1^i \,\mathrm{Bi}_1\, U_0\left(\mu_n \frac{R_1}{R}\right); \quad \mathrm{Bi}_1 = h_1 R.$$

Finally, permutation function $\Phi(\xi, \tau)$ can be also taken in the form

$$\Phi(\xi, \tau) = \psi_2(\tau) F_a(\xi) + \psi_1(\tau) F_b(\xi).$$

Therefore, the following expression is also correct for determination of the function $\theta(\xi, \tau)$:

$$\theta(\xi, \tau) = \sum_{n=1}^{\infty} \frac{1}{\| U_0 \|^2} [T_2 \bar{F}_a + T_1 \bar{F}_b] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Thus, the solution of problem (3-82)-(3-84) with constant temperature of the medium T_j = const (j = 1, 2) can be written as

$$T = T_1 + (T_2 - T_1) F_a(\xi) - R \sum_{n=1}^{\infty} \frac{1}{\mu_n^2 \| U_0 \|^2} \left[R_2^i \operatorname{Bi}_2 T_2 U_0\left(\mu_n \frac{R_2}{R}\right) + \right.$$
$$\left. + R_1^i \operatorname{Bi}_1 T_1 U_0\left(\mu_n \frac{R_1}{R}\right)\right] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Problem 2. If the temperature of the medium depends on time as a polynomial of power m_j, so that

$$\psi_1(\tau) = \sum_{s=0}^{m_1} \psi_{1s} \tau^s; \quad \psi_2(\tau) = \sum_{s=0}^{m_2} \psi_{2s} \tau^s,$$

then, taking for function $\Phi(\xi, \tau)$ the form

$$\Phi(\xi, \tau) = \psi_2 F_a(\xi) + \psi_1 F_б(\xi),$$

we produce for determination of function $\theta(\xi, \tau)$ the equation

$$\frac{\partial \theta}{\partial \tau} = a \nabla^2 \theta + F_a(\xi) \sum_{s=0}^{m_2} s \psi_{2s} \tau^{s-1} + F_б(\xi) \sum_{s=0}^{m_1} s \psi_{1s} \tau^{s-1}$$

and the edge condition

$$\theta(\xi, 0) = \psi_{20} F_a(\xi) + \psi_{10} F_6(\xi);$$
$$\frac{\partial\theta(R_j, \tau)}{\partial\xi} + (-1)^j h_j \theta(R_j,)\tau = 0.$$

We have already solved similar problems, and effective methods have been suggested for summation of some of the series encountered in the solutions.

Thus, the desired temperature function is:

$$T = F_a(\xi) \sum_{s=0}^{m_2} \psi_{2s} \tau^s + F_6(\xi) \sum_{s=0}^{m_1} \psi_{1s} \tau^s - \theta.$$

Problem 3. Suppose

$$\psi_j = \psi_{j_0} e^{\mp m_j \tau} \ (j = 1, 2),$$

where m_j may be either real or imaginary[1].

We select for $\Phi(\xi, \tau)$ the form

$$\Phi(\xi, \tau) = \psi_2 F_a(\xi) + \psi_1 F_6(\xi).$$

Then, considering that $\psi_j' = \pm m_j \psi_j$, we produce

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta - a\psi_2\nabla^2 F_a - a\psi_1\nabla^2 F_6 \mp m_2\psi_2 F_a \mp m_1\psi_1 F_6.$$

It is natural to require that

$$\nabla^2 F_a = \mp \frac{m_2}{a} F_a; \ \nabla^2 F_6 = \mp \frac{m_1}{a} F_6.$$

Consequently, to determine function $\theta(\xi, \tau)$ we have the homogeneous equation

$$\frac{\partial\theta}{\partial\tau} = a\nabla^2\theta \ (R_1 < \xi < R_2, \tau > 0)$$

[1] This allows us to extend the results produced to cases when the temperature of the medium changes according to a cosine (sine) rule.

and the edge conditions

$$\theta(\xi, 0) = \psi_{20} F_a(\xi) + \psi_{10} F_б(\xi) \quad (R_1 \leq \xi \leq R_2);$$
$$\frac{\partial\theta(R_j, \tau)}{\partial\xi} + (-1)^j h_j \theta(R_j, \tau) = 0 \quad (j = 1, 2),$$

where F_a and F_b are the solutions of the following problems:

$$\frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i \frac{dF_a}{d\xi}\right) \pm \frac{m_2}{a} F_a = 0;$$
$$\frac{dF_a(R_1)}{d\xi} = h_1 F_a(R_1); \quad \frac{dF_a(R_2)}{d\xi} = h_2[1 - F_a(R_2)];$$
$$\frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i \frac{dF_б}{d\xi}\right) \pm \frac{m_1}{a} F_б = 0;$$
$$\frac{dF_б(R_1)}{d\xi} = -h_1[1 - F_б(R_1)]; \quad \frac{dF_б(R_2)}{d\xi} = -h_2 F_б(R_2).$$

The solution of the problem for function θ is obvious from the above. The final temperature function $T(\xi, \tau)$ is equal to:

$$T = \psi_{20} F_a(\xi) e^{-m_2\tau} + \psi_{10} F_б(\xi) e^{-m_1\tau} -$$
$$- \sum_{n=1}^{\infty} \frac{1}{\| U_0 \|^2} [\psi_{20}\bar{F}_a + \psi_{10}\bar{F}_б] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}},$$

where

$$\bar{F}_a = \int_{R_1}^{R_2} \xi^i F_a(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \frac{1}{\mu_n^2 \mp \frac{m_2}{a}} R R_2^i \mathrm{Bi}_2 U_0\left(\mu_n \frac{R_2}{R}\right);$$
$$\bar{F}_б = \int_{R_1}^{R_2} \xi^i F_б(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi = \frac{1}{\mu_n^2 \mp \frac{m_1}{a}} R R_1^i \mathrm{Bi}_1 U_0\left(\mu_n \frac{R_1}{R}\right).$$

Let us introduce the following symbols, which we will continue to use. Functions $F(\xi)$ satisfying the equation

$$\nabla^2 F = 0, \tag{3-96}$$

will be marked with the subscript I ($F = F_I(\xi)$ and be called F functions of the first kind, while functions satisfying the equation

$$\nabla^2 F \pm \frac{m}{a} F = 0, \tag{3-97}$$

will be marked with the subscript II ($F = F_{II}(\xi)$) and called F functions of the second kind.

Depending on whether function $F(\xi)$ satisfies boundary condition "a" or "b" (see tables 3-1 and 3-2), we will add the latter "a" or "b" to the subscript. Thus, for example, a function $F_{Ia}(\xi)$ satisfies equation (3-96) and boundary condition "a." The results produced concerning selection and determination of functions $F_I(\xi)$ and $F_{II}(\xi)$ extend almost fully to problems with boundary conditions of the first, second and third kind and to those with mixed boundary conditions. The exception is an edge problem with boundary conditions of the second kind. In this case, we must require that

$$\nabla^2 F_I = \text{const}$$

and

$$\int_{R_1}^{R_2} \xi^i F_I(\xi)\, d\xi = 0,$$

since

$$\nabla^2 \Phi = \frac{2i}{\lambda (R_2^i + R_1^i)(R_2 - R_1)} [R_1^i \eta_1(\tau) + R_2^i \eta_2(\tau)].$$

Thus, in order to improve the convergence of series produced in solution of problems of heat conductivity with heterogeneous boundary conditions, the permutation functions $\Phi(\xi, \tau)$ are effective (see tables 3-1 and 3-2).

The form of the functions $F(\xi)$ included in the expression for $\phi(\xi, \tau)$ is established on the basis of the following considerations.

If the time dependence of the ambient temperature, surface temperature or heat flux to the surface of the body can be represented in the form of a polynomial (including a constant), then

$$\nabla^2 F_I = \frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i \frac{dF_I}{d\xi}\right) = \begin{cases} \text{const}^{1} & \text{with boundary conditions of 2nd kind,} \\ 0 & \text{in all other cases} \end{cases} \tag{3-98}$$

If the time dependence of the ambient temperature, surface temperature or heat flux to the surface of the body is represented by an exponential function, where the exponent may be real or imaginary, then

$$\nabla^2 F_{II} \pm \omega F_{II} = 0. \tag{3-99}$$

The boundary conditions for functions $F_I(\xi)$ and $F_{II}(\xi)$ are defined by the data in tables 3-1 and 3-2. Obviously, depending on the boundary conditions in the initial problem for construction of the function $\Phi(\xi, \tau)$ various combinations of functions $F_I(\xi)$ and $F_{II}(\xi)$ are possible.

Supplements. The considerations presented above concerning the use of permutation functions in the solution of heat conductivity problems by the method of G. A. Greenberg concerned those cases when the heterogeneity $g_j(\tau)$ in the boundary conditions of the general form

$$\alpha_j \frac{\partial T(R_j, \tau)}{\partial \xi} + \beta_j T(R_j, \tau) = \gamma_j g_j(\tau) \quad (j = 1, 2) \tag{3-100}$$

were smooth functions of time.

Let us extend this method to any piecewise-smooth functions $g_j(\tau)$[2].

Suppose for definition function $g(\tau)$[3] is equal to

---

[1]Here

$$\int_{R_1}^{R_2} \xi^i F_I(\xi)\, d\xi = 0$$

and

$$\nabla^2 \Phi = \frac{2i}{\lambda (R_2^i + R_1^i)(R_2 - R_1)} (R_1^i \eta_1 + R_2^i \eta_2).$$

[2]We recall that smooth functions refer to those which are continuous in the main area $[R_1, R_2]$, having continuous first derivatives. Piecewise-smooth functions are functions with piecewise-continuous first derivatives.

[3]For simplicity, we will omit the symbol j.

$$g(\tau)=\begin{cases}\Delta g_0+g_1(\tau-\tau_0) \text{ при } \tau_0=0<\tau<\tau_1;\\ \Delta g_0+g_1(\tau_1-\tau_0)+\Delta g_1+g_2(\tau-\tau_1) \text{ при } \tau_1<\tau<\tau_2;\\ \Delta g_0+g_1(\tau_1-\tau_0)+\Delta g_1+g_2(\tau_2-\tau_1)+\Delta g_2+g_3(\tau-\tau_2) \text{ при } \tau>\tau_2,\end{cases} \tag{3-101}$$

where Δg_k is the jump in function g at point $\tau = \tau_k$.

Let us introduce the unit function

$$e(\tau-\tau_k)=\begin{cases}0 \text{ where } \tau-\tau_k<0;\\ 1 \text{ where } \tau-\tau_k>0\end{cases}$$

and write the expression (3-101) as follows:

$$g(\tau)=\Delta g_0+g_1(\tau-\tau_1)+[-g_1(\tau-\tau_0)+g_1(\tau_1-\tau_0)+ \\ +\Delta g_1+g_2(\tau-\tau_1)]\,e(\tau-\tau_1)+[-g_2(\tau-\tau_1)+ \\ +g_2(\tau_2-\tau_1)+\Delta g_2+g_3(\tau-\tau_2)]\,e(\tau-\tau_2).$$

Then the derivative

$$g'(\tau)=g'_1(\tau-\tau_0)+[-g'_1(\tau-\tau_0)+g'_2(\tau-\tau_1)]\,e(\tau-\tau_1)+ \\ +[-g_1(\tau-\tau_0)+g_1(\tau_1-\tau_0)+\Delta g_1+g_2(\tau-\tau_1)]\,\delta(\tau-\tau_1)+ \\ +[-g'_2(\tau-\tau_1)+g'_3(\tau-\tau_2)]\,e(\tau-\tau_2)+[-g_2(\tau-\tau_1)+ \\ +g_2(\tau_2-\tau_1)+\Delta g_2+g_3(\tau-\tau_2)]\,\delta(\tau-\tau_2),$$

where $\delta(\tau - \tau_k)$ is a delta function (for more detail concerning delta functions see § 3-5).

This representation of function $g(\tau)$ and its derivative $g'(\tau)$ allows us to use permutation functions in correspondence with the recommendations given earlier. We should keep in mind here the basic properties of delta functions and unit functions presented in § 3-5.

As an illustration, let us study the solution of the following problem:

$$\frac{\partial T}{\partial \tau} = a\,\frac{\partial^2 T}{\partial x^2}\quad (0 < x < R,\ \tau > 0);$$

$$T(x, 0) = 0\quad (0 \leqslant x \leqslant R);$$

$$\frac{\partial T(0, \tau)}{\partial x} = 0;\quad \frac{\partial T(R, \tau)}{\partial x} = h\,[\psi(\tau) - T(R, \tau)],$$

where

$$\psi(\tau) = \begin{cases} \Delta T_0 \text{ where } \tau_0 = 0 < \tau < \tau_1; \\ \Delta T_0 + b_2(\tau - \tau_1)^{1} \text{ where } \tau_1 < \tau < \tau_2; \\ \Delta T_0 + b_2(\tau_2 - \tau_1) + \Delta T_2 \text{ where } \tau > \tau_2. \end{cases}$$

Let us write function $\psi(\tau)$

$$\psi(\tau) = \Delta T_0 + b_2(\tau - \tau_1)\,e(\tau - \tau_1) + [-b_2(\tau - \tau_2) + \Delta T_2]\,e(\tau - \tau_2).$$

The derivative $\psi'(\tau)$ is:

$$\psi'(\tau) = b_2 e(\tau - \tau_1) + b_2(\tau - \tau_1)\,\delta(\tau - \tau_1) - b_2 e(\tau - \tau_2) +$$
$$+ [-b_2(\tau - \tau_2) + \Delta T_2]\,\delta(\tau - \tau_2).$$

We assume:

$$T = \Phi(\xi, \tau) - \theta(x, \tau),$$

where according to the data of Table 3-2

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[1]Here in contrast to the previous

$$b_2(\tau - \tau_k) = b_2 \times (\tau - \tau_k).$$

$$\Phi(x, \tau) = \psi(\tau).$$

Then

$$\frac{\partial \theta}{\partial \tau} = a\frac{\partial^2 \theta}{\partial x^2} + \psi'(\tau) \quad (0 < x < R,\ \tau > 0);$$

$$\theta(x, 0) = \psi(0) \quad (0 \leqslant x \leqslant R);$$

$$\frac{\partial \theta(0, \tau)}{\partial x} = 0; \quad \frac{\partial T(R, \tau)}{\partial x} = -hT(R, \tau).$$

The corresponding Shturm-Liouville problem

$$\frac{d^2 U_0}{dx^2} + \frac{\mu_n^2}{R^2} U_0 = 0 \quad (0 < x < R);$$

$$\frac{dU_0(0)}{dx} = 0; \quad \frac{dU_0(R)}{dx} = -hU_0(R).$$

Therefore, the finite integral transform of the problem for function θ is defined by the formulas

$$\bar{\theta}_n = \int_0^R \theta U_0\left(\mu_n \frac{x}{R}\right) dx;$$

$$\theta = \sum_{n=1}^{\infty} \frac{1}{\| U_0 \|^2} \bar{\theta}_n U_0\left(\mu_n \frac{x}{R}\right),$$

where $U_0(\mu_n \frac{x}{R}) = \cos \mu_n \frac{x}{R}$ is the Eigenfunction; μ_n is the root of the characteristic equation

$$\operatorname{ctg} \mu_n = \frac{\mu_n}{\mathrm{Bi}}; \quad \mathrm{Bi} = hR;$$

$$\| U_0 \|^2 = \frac{R}{2} \frac{\mu_n^2 + \mathrm{Bi}^2 + \mathrm{Bi}}{\mu_n^2 + \mathrm{Bi}^2}.$$

Performing a transform, we come to the equation

$$\frac{d\bar{\theta}_n}{d\tau} + \frac{a\mu_n^2}{R^2} \bar{\theta}_n - \psi'(\tau) N_n = 0$$

and the initial condition

$$\bar{\theta}_n(0) = \psi(0) N_n,$$

where

$$N_n = \int_0^R U_0\left(\mu_n \frac{x}{R}\right) dx = \frac{R}{\mu_n} \frac{(-1)^{n+1} \mathrm{Bi}}{\sqrt{\mu_n^2 + \mathrm{Bi}^2}}.$$

From this

$$\bar{\theta}_n = \psi(0) N_n e^{-\mu_n^2 \frac{a\tau}{R^2}} + N_n e^{-\mu_n^2 \frac{a\tau}{R^2}} \int_0^\tau \psi'(\tau) e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau$$

and

$$T = \psi(\tau) - \psi(0) \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}} -$$

$$- \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}} \int_0^\tau \psi'(\tau) e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau,$$

where

$$A_n = \frac{N_n}{\| U_0 \|^2} = \frac{(-1)^{n+1} 2 \mathrm{Bi} \sqrt{\mu_n^2 + \mathrm{Bi}^2}}{\mu_n (\mu_n^2 + \mathrm{Bi}^2 + \mathrm{Bi})}.$$

Consequently, the final solution is:

1. $0 < \tau < \tau_1$.

The integral

$$I = \int_0^\tau \psi'(\tau) e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau = 0.$$

The temperature function

$$T = \Delta T_0 - \Delta T_0 \sum_{n=1}^{\infty} A_n \cos \mu_n \frac{x}{R} e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

2. $\tau_1 < \tau < \tau_2$.

The integral

$$I = \int_0^{\tau} \psi'(\tau) e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau = b_2 \int_{\tau_1}^{\tau} e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau + b_2 \int_0^{\tau} (\tau - \tau_1) \times$$
$$\times e^{\mu_n^2 \frac{a\tau}{R^2}} \delta(\tau - \tau_1) d\tau = \frac{b_2 R^2}{a\mu_n^2} \left[e^{\mu_n^2 \frac{a\tau}{R^2}} - e^{\mu_n^2 \frac{a\tau_1}{R^2}} \right].$$

The temperature function

$$T = \Delta T_0 + b_2 (\tau - \tau_1) - \sum_{n=1}^{\infty} A_n \left[\Delta T_0 e^{-\mu_n^2 \frac{a\tau}{R^2}} - \right.$$
$$\left. - \frac{b_2 R^2}{a\mu_n^2} e^{-\mu_n^2 \frac{a(\tau - \tau_1)}{R^2}} \right] \cos \mu_n \frac{x}{R} - \frac{b_2 R^2}{a} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} \cos \mu_n \frac{x}{R}.$$

3. $\tau > \tau_2$.

The integral

$$I = \int_0^{\tau} \psi'(\tau) e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau = b_2 \int_0^{\tau_1} e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau + b_2 \int_0^{\tau} (\tau - \tau_1) \times$$
$$\times e^{\mu_n^2 \frac{a\tau}{R^2}} \delta(\tau - \tau_1) d\tau - b_2 \int_0^{\tau_2} e^{\mu_n^2 \frac{a\tau}{R^2}} d\tau - b_2 \int_0^{\tau} (\tau - \tau_2) \times$$
$$\times e^{\mu_n^2 \frac{a\tau}{R^2}} \delta(\tau - \tau_2) d\tau + \Delta T_2 \int_0^{\tau} e^{\mu_n^2 \frac{a\tau}{R^2}} \delta(\tau - \tau_2) d\tau.$$

---

[1] Here we consider the primary property of delta function

$$\int_0^{\tau} f(\tau) \delta(\tau - \tau_k) d\tau = f(\tau_k).$$

The temperature function

$$T = \Delta T_0 + b_2(\tau_2 - \tau_1) + \Delta T_2 - \sum_{n=1}^{\infty} A_n \left[\Delta T_0 e^{-\mu_n^2 \frac{a\tau}{R^2}} - \right.$$
$$\left. - \frac{b_2 R^2}{a\mu_n^2} e^{-\mu_n^2 \frac{a(\tau-\tau_1)}{R^2}} + \left(\frac{b_2 R^2}{a\mu_n^2} + \Delta T_2\right) e^{-\mu_n^2 \frac{a(\tau-\tau_2)}{R^2}} \right] \cos \mu_n \frac{x}{R}.$$

One series in these formulas is summated. As follows from the preceding one, its sum is:

$$\sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} \cos \mu_n \frac{x}{R} = 2\left(1 + \frac{2}{\mathrm{Bi}} - \frac{x^2}{R^2}\right).$$

Example 1. Establish the temperature field of a symmetrically heated concrete wall of width 2R, the external surface of which is maintained at a temperature which changes exponentially with time. The initial temperature of the wall is equal to 0.

Due to the symmetry of the temperature field, we place the coordinate origin at the center of the wall and analyze the area $0 \leq x \leq R$.

Then the problem is formulated mathematically as follows:

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2} \quad (0 < x < R, \ \tau > 0);$$
$$T(x, 0) = 0 \quad (0 \leqslant x \leqslant R);$$
$$\frac{\partial T(0, \tau)}{\partial x} = 0, \quad T(R, \tau) = T_0 e^{-m\tau}.$$

The Eigenfunction of the problem

$$U_0\left(\mu_n \frac{x}{R}\right) = \cos \mu_n \frac{x}{R},$$

where μ_n is the root of the characteristic equation

$$\cos \mu_n = 0,$$

i.e.,

$$\mu_n = (2n-1)\frac{\pi}{2} \quad (n = 1, 2\ldots);$$

$$\| U_0 \|^2 = \frac{R}{2}; \quad N_n = \frac{(-1)^{n+1} 2R}{(2n-1)\pi}.$$

The solution will be produced by two methods.

1. Direct application of an integral transform yields:

$$\frac{d\bar{T}_n}{d\tau} + \frac{\mu_n^2 a}{R^2}\bar{T}_n - (-1)^{n+1}\frac{a\mu_n}{R} T_0 e^{-m\tau} = 0;$$
$$\bar{T}_n(0) = 0.$$

From this

$$\bar{T}_n = (-1)^{n+1} R T_0 \frac{\mu_n}{\mu_n^2 - m^{*2}} [\exp(-m\tau) - \exp(-\mu_n^2 \text{Fo})],$$

where

$$m^{*2} = \frac{mR}{a}; \quad \text{Fo} = \frac{a\tau}{R^2},$$

and the temperature function $T(x, \tau)$ is equal to:

$$T = T_0 e^{-m\tau} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)\pi}{(2n-1)^2 \frac{\pi^2}{4} - m^{*2}} \cos\frac{(2n-1)\pi}{2}\frac{x}{R} -$$
$$- T_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)\pi}{(2n-1)^2 \frac{\pi^2}{4} - m^{*2}} \cos\frac{(2n-1)\pi}{2}\frac{x}{R} \times$$
$$\times \exp\left[-\frac{(2n-1)^2\pi^2}{4}\text{Fo}\right]. \tag{3-102}$$

2. We assume

$$T = T_0 e^{-m\tau} - \theta.$$

Then

$$\frac{\partial\theta}{\partial\tau} = a\frac{\partial^2\theta}{\partial x^2} - mT_0e^{-m\tau} \quad (0 < x < R,\ \tau > 0);$$

$$\theta(x, 0) = T_0 \quad (0 \leqslant x \leqslant R);$$

$$\frac{\partial\theta(0, \tau)}{\partial x} = 0; \quad \theta(R, \tau) = 0.$$

We perform an integral transform.

We have:

$$\frac{d\overline{\theta}_n}{d\tau} + \frac{\mu_n^2 a}{R^2}\overline{\theta}_n + mT_0e^{-m\tau}N_n = 0;$$

$$\overline{\theta}_n(0) = T_0N_n,$$

from which

$$\overline{\theta}_n = m^{*2}T_0\frac{N_n}{\mu_n^2 - m^{*2}}e^{-m\tau} + T_0\frac{\mu_n^2 N_n}{\mu_n^2 - m^{*2}}e^{-\mu_n^2\frac{a\tau}{R^2}}.$$

Consequently,

$$T = T_0e^{-m\tau} - 4m^{*2}T_0e^{-m\tau}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\frac{(2n-1)\pi}{2}\frac{x}{R}}{(2n-1)\pi\left[(2n-1)^2\frac{\pi^2}{4} - m^{*2}\right]} -$$

$$-T_0\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)\pi}{\left[(2n-1)^2\frac{\pi^2}{4} - m^{*2}\right]}\cos\frac{(2n-1)\pi}{2}\frac{x}{R}\times$$

$$\times\exp\left[-\frac{(2n-1)^2\pi^2}{4}\frac{a\tau}{R^2}\right]. \qquad (3\text{-}103)$$

As we can see from comparison of the solutions of the problem in form (3-102) and (3-103), the permutation function $\Phi = T_0e^{-m\tau}$, although it did not fully segregate the summable portion of the solution, did significantly improve the convergence of the series. For example, the first series in formulas (3-102), produced as a result of direct application of the integral transforms, converges as $1/(2n - 1)$, whereas in formula (3-103), produced by the same

method but using the permutation function, the order of convergence is $1/(2n - 1)^3$.

We note that frequently the representation of the solution in the form of a rather rapidly converging series is preferable to cumbersome summation formulas. However, this is not true of the simple example here in question.

It is not difficult to see from the preceding results that the sum of the first series in formula (3-102) is the solution to the problem

$$\frac{dw}{dx^2} + \frac{m}{a} w = 0 \quad (0 < x < R);$$
$$\frac{dw(0)}{dx} = 0, \quad w(R) = 1,$$

i.e.,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)\pi}{\left[(2n-1)^2 \frac{\pi^2}{4} - m^{*2}\right]} \cos \frac{(2n-1)\pi}{2} \frac{x}{R} = \frac{\cos m^* \frac{x}{R}}{\cos m^*}.$$

Similarly, we can add the series in solution (3-103), if we analyze the problem

$$\frac{d^2w}{dx^2} + \frac{m}{a} w = -1 \quad (0 < x < R);$$
$$\frac{dw(0)}{dx} = 0, \quad w(R) = 0.$$

Example 2. The temperature of one surface of a body ($\xi = R_2$) changes with time according to a cosine rule, while on the other surface of the body ($\xi = R_1$) boundary conditions of the third kind are fixed, the ambient temperature increases linearly. The initial temperature of the body is a function of the coordinates. Establish the temperature field of the body.

The mathematical formulation of the problem

$$\frac{\partial T}{\partial \tau}=\frac{1}{\xi^s}\frac{\partial}{\partial \xi}\left(\xi^s\frac{\partial T}{\partial \xi}\right)\quad (R_1<\xi<R_2,\ \tau>0,\ s=0\vee 1)^1;$$
$$T(\xi, 0)=f(\xi)\quad (R_1\leqslant \xi\leqslant R_2);$$
$$\frac{\partial T(R_1, \tau)}{\partial \xi}=-h_1[\psi_1(\tau)-T(R_1, \tau)];$$
$$T(R_2, \tau)=\varphi_2(\tau),$$

where

$$\psi_1(\tau)=b\tau;$$
$$\varphi_2(\tau)=A\cos\omega\tau=\operatorname{Re}Ae^{i\omega\tau}.$$

We assume

$$T(\xi, \tau)=\Phi(\xi, \tau)-\theta(\xi, \tau),$$

where $\Phi(\xi, \tau)$ is the permutation function, which is selected as

$$\Phi(\xi, \tau)=\varphi_2 F_{IIa}(\xi)+\psi_1 F_{Ib}(\xi).$$

The form of the functions $F_{IIa}(\xi)$ and $F_{Ib}(\xi)$ is established from the data of tables 3-1 and 3-2, though in constructing function $F_{IIa}(\xi)$ we must consider that the exponent in the formula for the ambient temperature is imaginary.

We recall:

$$\frac{1}{\xi^s}\frac{d}{d\xi}\left(\xi^s\frac{dF_{IIa}}{d\xi}\right)-i\frac{\omega}{a}F_{IIa}=0;$$
$$\frac{dF_{IIa}(R_1)}{d\xi}=h_1F_{IIa}(R_1);\quad F_{IIa}(R_2)=1$$

and

[1]Here, instead of the usual Latin letter i, we use the letter s, in order to distinguish it from the imaginary unit i.

$$\frac{1}{\xi^s}\frac{d}{d\xi}\left(\xi^s\frac{dF_{I\text{б}}}{d\xi}\right)=0;$$

$$\frac{dF_{I\text{б}}(R_1)}{d\xi}=-h_1\left[1-F_{I\text{б}}(R_1)\right];\quad F_{I\text{б}}(R_2)=0.$$

Then

$$\frac{\partial\theta}{\partial\tau}=a\frac{1}{\xi^s}\frac{\partial}{\partial\xi}\left(\xi^s\frac{\partial\theta}{\partial\xi}\right)+bF_{I\text{б}}(\xi)\quad(R_1<\xi<R_2,\ \tau>0,\ s=0\vee 1);$$

$$\theta(\xi,0)=AF_{IIa}(\xi)-f(\xi)\quad(R_1\leqslant\xi\leqslant R_2);$$

$$\frac{\partial\theta(R_1,\tau)}{\partial\xi}=h_1\theta(R_1,\tau);\quad\theta(R_2,\tau)=0$$

and after the integral transform

$$\frac{d\bar\theta_n}{d\tau}+\mu_n^2\frac{a}{R^2}\bar\theta_n-b\bar F_{I\text{б}}=0;$$

$$\bar\theta_n(0)=A\bar F_{IIa}-\bar f_n.$$

Consequently,

$$\bar\theta_n=(A\bar F_{IIa}-\bar f_n)\,e^{-\mu_n^2\frac{a\tau}{R^2}}+\frac{bR^2}{a\mu_n^2}F_{I\text{б}}\left(1-e^{-\mu_n^2\frac{a\tau}{R^2}}\right).$$

Thus, the desired temperature function $T(\xi,\tau)$ is:

$$T=\mathrm{Re}\,Ae^{i\omega\tau}F_{IIa}(\xi)+b\tau F_{I\text{б}}(\xi)-$$
$$-\frac{bR^2}{a}\sum_{n=1}^{\infty}\frac{\bar F_{I\text{б}}}{\mu_n^2\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)+\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\left(\bar f_n+\frac{bR^2}{a}\frac{\bar F_{I\text{б}}}{\mu_n^2}\right)\times$$
$$\times U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}}-\mathrm{Re}\,A\sum_{n=1}^{\infty}\frac{\bar F_{IIa}}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}}.\qquad(3\text{-}104)$$

The first series in formula (3-104) is summated, its sum is the solution to the problem

$$\frac{1}{\xi^4}\frac{d}{d\xi}\left(\xi^4\frac{dw}{d\xi}\right)=-F_{1b}(\xi)\ (R_1<\xi<R_2);$$

$$\frac{dw(R_1)}{d\xi}=h_1w(R_1);\ w(R_2)=0.$$

3-4. The Laplace Transform

The Direct Laplace Transform

The direct Laplace transform of function $u(\tau)$ of real variable τ is defined by the formula

$$\mathcal{L}[u(\tau)]=\bar{u}(s)=\int_0^\infty e^{-s\tau}u(t)\,dt. \qquad (3\text{-}105)$$

Here $\mathcal{L}$ is the operator of the direct Laplace transform.

Integral (3-105) is a singular integral, depending on the complex variable $s = \eta + i\omega$ as on a parameter.

If function $u(\tau)$ where $\tau < 0$ is equal to 0, where $\tau \geq 0$ is piecewise-continuous over any open interval (0, A) and with all $\tau > 0$ the modulus of the function

$$|u(\tau)| \leq Me^{a\tau} \text{ (M and a are positive constants)}, \qquad (3\text{-}105')$$

then integral (3-105) converges in the area Re $s > a$, and mapping $\bar{u}(s)$ is an analytic function of complex variable s in this area; here, in the area Re $s \geq \sigma_0 > a$, the integral converges absolutely and evenly.

For each function $u(\tau)$ satisfying the conditions presented, the Laplace transform is unique.

Definition 1. Function $\bar{u}(s)$ of complex variable $s = \eta + i\omega$ is called analytic in area D if it is differentiable in each point of this area.

Definition 2. The maximum value of real numbers a, for which inequality (3-105') obtains, is called the growth exponent of function $u(\tau)$.

Definition 3. The minimum value σ_a of real numbers σ_0 for which the integral

$$\int_0^\infty |u(t)| e^{-at} dt$$

converges is called the abscissa of the absolute convergence of Laplace transform (3-105).

Basic properties of the Laplace transform.

1. The property of linearity

$$\mathcal{L}[\alpha u(\tau) + \beta g(\tau)] = \alpha \mathcal{L}[u(\tau)] + \beta \mathcal{L}[g(\tau)] = \\ = \alpha \bar{u}(s) + \beta \bar{g}(s); \ \operatorname{Re} s > \max\{a, b\},$$

(α, β are constants, generally complex; a, b are the growth function exponents of $u(\tau)$ and $g(\tau)$).

2. The property of similarity

$$\mathcal{L}[u(\alpha\tau)] = \frac{1}{\alpha} \bar{u}\left(\frac{s}{\alpha}\right); \ \mathcal{L}\left[\frac{1}{\alpha} u\left(\frac{\tau}{\alpha}\right)\right] = \bar{u}(\alpha s), \\ \alpha > 0; \ \operatorname{Re} s > a.$$

3. The bias theorem

$$\mathcal{L}[e^{-\lambda\tau} u(\tau)] = \bar{u}(s+\lambda); \ \operatorname{Re} s > a - \operatorname{Re} \lambda.$$

4. The delay theorem. Suppose

$$u_b(\tau) = \begin{cases} 0, & 0 < \tau < b, \\ u(\tau - b), & \tau > b. \end{cases}$$

Then

$$\mathcal{L}[u_b(\tau)] = e^{-sb} \bar{u}(s); \ \operatorname{Re} s > a.$$

5. The mapping of the derivative

$$\mathcal{L}\left[\frac{d^{(n)}u(\tau)}{d\tau^{(n)}}\right]=s^n\bar{u}(s)-s^{n-1}u(0)-$$

$$-s^{n-2}u'(0)-\ldots-u^{(n-1)}(0)\ (n=1,2,\ldots);\ \operatorname{Re} s>a$$

[if $u^{(n)}(\tau)$ exists with all $\tau > 0$];

in particular

$$\mathcal{L}\left[\frac{du(\tau)}{d\tau}\right]=s\bar{u}(s)-u(0).$$

6. The mapping of the integral

$$\mathcal{L}\left[\int_0^\tau u(t)\,dt\right]=\frac{\bar{u}(s)}{s};\ \operatorname{Re} s>a.$$

7. Differentiation of the mapping

$$\frac{d^{(n)}\bar{u}(s)}{ds^{(n)}}=\mathcal{L}\left[(-\tau)^n u(\tau)\right];\ \operatorname{Re} s>a.$$

8. Integration of the mapping

$$\int_s^\infty \bar{u}(s)\,ds=\mathcal{L}\left[\frac{u(\tau)}{\tau}\right];\ \operatorname{Re} s>a$$

[it is assumed that the integral $\int_s^\infty u(s)ds$ converges].

9. The Borel convolution theorem.

Definition. Convolution of functions $u(\tau)$ and $g(\tau)$ in interval $(0, \tau)$ refers to the function

$$u(\tau)*g(\tau)=\int_0^\tau u(t)\,g(\tau-t)\,dt=\int_0^\tau u(\tau-t)\,g(t)\,dt.$$

Theorem.

a) $\mathcal{L}[u(\tau)]\mathcal{L}[g(\tau)] = \mathcal{L}\left[\int_0^\tau u(t)\,g(\tau - t)\,dt\right] = \mathcal{L}\left[\int_0^\tau u(\tau - t)\,g(t)\,dt\right]$

or more briefly

$$\bar{u}(s)\,\bar{g}(s) = \mathcal{L}[u(\tau) * g(\tau)];\ \mathrm{Re}\, s > \max\{a, b\}.$$

b) $\mathcal{L}^{-1}[\bar{u}(s)\,\bar{g}(s)] = u(\tau) * g(\tau).$

Note. It is assumed that the convolution $u(\tau)*g(\tau)$ exists; a sufficient condition for convergence of $\mathcal{L}[u(\tau)*g(\tau)]$ is absolute convergence of $\bar{u}(s)$ and $\bar{g}(s)$. $\mathcal{L}^{-1}$ is the operator of the inverse Laplace transform.

Inverse Laplace Transform

If function $\bar{u}(s)$ is a transform (mapping) of piecewise-smooth function $u(\tau)$, equal to 0 where $\tau < 0$ and having growth exponent a, then at any point in its continuity, function $u(\tau)$ (the original) is equal to:

$$u(\tau) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\tau}\,\bar{u}(s)\,ds \tag{3-106}$$

(the Riemann-Mellin inversion formula), where the integral is taken along any straight line $\mathrm{Re}\, s = \sigma > a$.

Since $\bar{u}(s)$ is an analytic function in the area $\mathrm{Re}\, s > a$, the integration line in formula (3-106) should be selected to the right of all singular points[1] of function $u(s)$.

One sufficient condition for the existence of an original is: if $\bar{u}(s)$ is an analytic function in half plane $\mathrm{Re}\, s > a$, approaches 0 as $|s| \to \infty$ evenly relative to arg s in the area $\mathrm{Re}\, s > a$ and for all $\mathrm{Re}\, s = \sigma > a$ converges

---

[1] We have here in mind an isolated singular point. An isolated singular point of function $u(s)$ refers to a point s_0 in the ε area of which (with the exception of point s_0 itself) $\bar{u}(s)$ is analytic. There are three types of singular points:
1) a singular point which can be eliminated [if there is a finite limit $\lim\limits_{s \to s_0} \bar{u}(s)$]; 2) a pole [if $\lim\limits_{s \to s_0} \bar{u}(s) = \infty$]; 3) an essentially singular point [if there is no limit $\lim\limits_{s \to s_0} \bar{u}(s)$].

$$\int_{\sigma-i\infty}^{\sigma+i\infty} |\bar{u}(s)|\, ds < M; \ \sigma > a,$$

then function $\bar{u}(s)$ where Re $s > a$ is a mapping of function $u(\tau)$, defined by the Riemann-Mellin inversion formula

$$u(\tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\tau}\, \bar{u}(s)\, ds; \ \sigma > a.$$

The original $u(\tau)$ is defined uniquely by the Laplace transform for almost all[1] $\tau > 0$.

Note. For most practical purposes, the agreement between functions $u(\tau)$ and $\bar{u}(s)$ is mutually unambiguous.

Methods of Finding the Original from the Mapping

The most general method of finding the original function from an assigned mapping is to apply the Riemann-Mellin inversion formula. However, it is frequently more convenient to use relatively simple methods, suitable for determination of classes of mapping functions.

1. Tabular method. Tables have been published in the literature containing both direct and inverse Laplace transforms (see [8, 44-46, 68-69]). When they are used, the problem is reduced to one of finding the mapping function $\bar{u}(s)$ and determining the corresponding original $u(\tau)$. If the desired mapping is not to be found in the table, one must attempt to find an original based on the mappings presented in the table and the properties of the Laplace transform outlined above.

2. First theorem of expansion. Suppose the mapping $\bar{u}(s)$ is a rational fraction function such as

$$\bar{u}(s) = \frac{\varphi(s)}{\psi(s)}, \tag{3-107}$$

where $\phi(s)$ and $\psi(s)$ are polynomials, equal to $\phi(s) = b_0 + b_1 s + \ldots + b_{m-1}s^{m-1} + b_m s^m$; $\psi(s) = c_0 + c_1 s + \ldots + c_{n-1}s^{n-1} + c_n s^n$ where the power of polynomial $\psi(s)$ is higher than the power of polynomial $\phi(s)$, i.e., $n > m$.

---

[1] With the exception, possibly, of a finite number of points (for example, discontinuities of the first kind, i.e., finite discontinuities).

Then:

a) If all roots s_j ($j = 1, 2, \ldots, n$) of polynomial $\psi(s)$ are simple, so that $\psi(s)$ equal $c_n(s - s_1)(s - s_2)\ldots(s - s_{n-1})(s - s_n)$, c_n is constant, then

$$u(\tau) = \mathcal{L}^{-1}[\bar{u}(s)] = \mathcal{L}^{-1}\left[\frac{\varphi(s)}{\psi(s)}\right] = \sum_{j=1}^{n} \frac{\varphi(s_j)}{\psi'(s_j)} e^{s_j\tau}; \tag{3-108}$$

b) If in the general case the root s_j ($j = 1, 2, \ldots, \ell$) has multiplicity m_j, so that

$$\psi(s) = c_n(s - s_1)^{m_1}(s - s_2)^{m_2}\ldots(s - s_{l-1})^{m_{l-1}}(s - s_l)^{m_l}$$
$$(m_1 + m_2 + \ldots + m_{l-1} + m_l = n),$$

then

$$u(\tau) = \mathcal{L}^{-1}[\bar{u}(s)] = \mathcal{L}^{-1}\left[\frac{\varphi(s)}{\psi(s)}\right] =$$
$$= \sum_{j=1}^{l} \frac{1}{(m_j - 1)!} \lim_{s \to s_j}\left\{\frac{d^{m_j-1}}{ds^{m_j-1}}\left[\frac{\varphi(s)(s - s_j)^{m_j}}{\psi(s)} e^{s\tau}\right]\right. . \tag{3-109}$$

Formula (3-108) is a particular case of formula (3-109).

Note 1. Suppose

$$\bar{u}(s) = \frac{\varphi(s)}{s\psi_1(s)}, \tag{3-110}$$

where the power of polynomial $\phi(s)$ does not exceed the power of polynomial $\psi_1(s)$, and $\psi_1(s)$ has simple, non-zero roots.

Then

$$u(\tau) = \mathcal{L}^{-1}\left[\frac{\varphi(s)}{s\psi_1(s)}\right] = \frac{\varphi(0)}{\psi_1(0)} + \sum_{j=1}^{n-1} \frac{\varphi(s_j)}{s_j\psi'_1(s_j)} e^{s_j\tau} \tag{3-111}$$

[the sum is taken with respect to all roots of polynomial $\psi_1(s)$].

Note 2. The inverse transform of the rational fraction function can also be produced as a result of expansion of mapping function $\bar{u}(s)$ into elementary fractions:

a) If $\psi(s)$ has only simple roots s_j, so that $\psi(s) = c_n(s - s_1)\ldots(s - s_n)$, then

$$\bar{u}(s) = \frac{\varphi(s)}{\psi(s)} = \frac{A_1}{s - s_1} + \ldots + \frac{A_n}{s - s_n}$$

and

$$u(\tau) = \mathcal{L}^{-1}\left[\frac{\varphi(s)}{\psi(s)}\right] = A_1 e^{s_1\tau} + \ldots + A_n e^{s_n\tau};$$

b) If $\psi(s)$ has roots s_j of multiplicity m_ℓ, so that

$$\psi(s) = c_n(s - s_1)^{m_1} \ldots (s - s_l)^{m_l},$$

then

$$\bar{u}(s) = \frac{A_1}{s - s_1} + \frac{A_2}{(s - s_1)^2} + \cdots + \frac{A_{m_1}}{(s - s_1)^{m_1}} + \cdots + \frac{B_1}{(s - s_l)} + \\ + \frac{B_2}{(s - s_l)^2} + \ldots + \frac{B_{m_l}}{(s - s_l)^{m_l}}$$

and

$$u(\tau) = A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau} + \ldots + A_{m_1} \frac{\tau^{m_1 - 1}}{(m_1 - 1)!} e^{s_1\tau} + \cdots \\ \cdots + B_1 e^{s_l\tau} + B_2 \tau e^{s_l\tau} + \ldots + B_{m_l} \frac{\tau^{m_l - 1}}{(m_l - 1)!} e^{s_l\tau};$$

c) If among the roots of polynomial $\psi(s)$ there are complex single roots, so that

$$\psi(s) = c_n(s - s_1)^{m_1} \ldots (s - s_l)^{m_l}(s^2 + ps + q),$$

then

$$\bar{u}(s)=\frac{A_1}{s-s_1}+\ldots+\frac{A_{m_1}}{(s-s_1)^{m_1}}+\ldots+\frac{B_{m_l}}{(s-s_l)^{m_l}}+\frac{Bs+C}{s^2+ps+q}$$

and

$$u(\tau)=A_1e^{s_1\tau}+\ldots+A_{m_1}\frac{\tau^{m_1-1}}{(m_1-1)!}e^{s_1\tau}+\ldots+B_{m_l}\frac{\tau^{m_l-1}}{(m_l-1)!}e^{s_l\tau}+$$
$$+Be^{-\frac{p}{2}\tau}\left(\cos k\tau-\frac{p}{2k}\sin k\tau\right)+C\frac{1}{k}e^{-\frac{p}{2}\tau}\sin k\tau,$$

where

$$k^2=q-\frac{p^2}{4};$$

d) If among the roots of polynomial $\psi(s)$ there are multiple complex roots (particularly, for example, a double complex root), so that

$$\psi(s)=c_n(s-s_1)^{m_1}\ldots(s-s_l)^{m_l}(s^2+ps+q)^2,$$

then

$$\bar{u}(s)=\frac{A_1}{s-s_1}+\ldots+\frac{B_1s+C_1}{s^2+ps+q}+\frac{B_2s+C_2}{(s^2+ps+q)^2}$$

and

$$u(\tau)=A_1e^{s_1\tau}+\ldots+B_1e^{-\frac{p}{2}\tau}\left(\cos k\tau-\frac{p}{2k}\sin k\tau\right)+C_1\frac{1}{k}\times$$
$$\times e^{-\frac{p}{2}\tau}\sin k\tau+B_2\frac{1}{2k^3}e^{-\frac{p}{2}\tau}\left(k^2\tau\sin k\tau+\frac{p}{2}k\tau\cos k\tau-\frac{p}{2}\sin k\tau\right)+$$
$$+C_2\frac{1}{2k^3}e^{-\frac{p}{2}\tau}(\sin k\tau-k\tau\cos k\tau),$$

where

$$k^2=q-\frac{p^2}{4}.$$

The constants A_j, B_j, C_j are found by the method of indefinite coefficients.

3. Determination of the original from the Riemann-Mellin inversion formula. As was noted earlier, the most general means of calculating the original $u(\tau)$ from the mapping $\bar{u}(s)$ is the Riemann-Mellin integral

$$u(\tau) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\tau} \bar{u}(s)\, ds. \tag{3-112}$$

Integration by formula (3-112) is performed in a complex plane along any straight line Re $s = \sigma > a$, lying to the right of all singular points of function $\bar{u}(s)$.

Mapping $\bar{u}(s)$ is an analytic function. Therefore, in calculating integral (3-112), one can use the well-known methods of the theory of analytic functions. They include primarily a change in the integration path using the Cauchy integral theorem or the subtraction theorem [63, 113].

Before going over to examples of calculation of integral (3-112), let us present the important Jordan lemma and the second theorem of expansion.

The Jordan lemma. If function $f(s)$ in the set of arcs C_R ($|s| = R$, Re $s > \sigma$, $R_0 < R < \infty$, Figure 3-1), where $R \to \infty$ trends evenly toward 0 relative to arg s, then for any positive $\tau > 0$

$$\lim_{R\to\infty} \int_{C_R} f(s)\, e^{s\tau}\, ds = 0. \tag{3-113}$$

Note 1. The condition of even trend of $f(s)$ toward 0 relative to arg s means that where $|s| = R$

$$|f(s)| < M_R,$$

where $M_R \to 0$ as $R \to \infty$.

Note 2. The set of arcs C_R in the lemma can be replaced with a sequence of arcs C_{R_n} ($|s| = R_n$, $n = 1, 2, \ldots$) such that as $R_n \to \infty$, none of the arcs intersects a singular point of function $f(s)$.

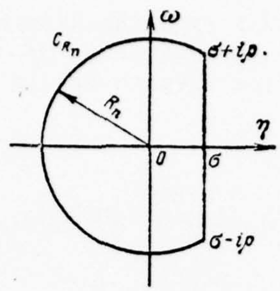

Figure 3-1. Integration Contour for Jordan Lemma

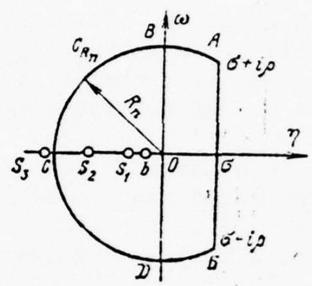

Figure 3-2. Integration Contour for Calculation of Inverse Laplace Transform of Unambiguous Functions

Second expansion theorem. If function $\bar{u}(s)$ is meromorphic[1] and analytic in a certain half plane Re s > σ_0 (σ_0 is constant), satisfies the conditions of the Jordan lemma, the following integral absolutely converges for any $\sigma > \sigma_0$ the original $u(\tau)$ is:

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{u}(s)\, ds,$$

original $u(\tau)$ is:

$$u(\tau) = \sum_{s_j} \operatorname{res} \bar{u}(s_j)\, e^{s_j \tau}, \tag{3-114}$$

where res $\bar{u}(s_j)$ is the residue of function $\bar{u}(s)$ at the singular point s_j, the sum of residues is taken with respect to all singular points in the sequence of nondecreasing moduli.

[1]Function f(s) is called meromorphic (fractional) in area D if the only singular points in this area are the poles.

Note 1. The first theorem of expansion is a result of the second theorem of expansion.

Note 2. The second expansion theorem is effectively used in those cases when function $\bar{u}(s)$ in the vicinity of singular point s_j is represented as a ratio of two analytic functions $\phi(s)$ and $\psi(s)$, i.e.

$$\bar{u}(s)=\frac{\varphi(s)}{\psi(s)}, \text{ while } \varphi(s_j)\neq 0.$$

If at point $s = s_j$ function $\psi(s)$ has:

a) a null of first order, then

$$\operatorname{res}\bar{u}(s_j)e^{s_j\tau}=\frac{\varphi(s_j)}{\psi'(s_j)}e^{s_j\tau};$$

b) a null of order m[1], then

$$\operatorname{res}\bar{u}(s_j)e^{s_j\tau}=\frac{1}{(m-1)!}\lim\frac{d^{m-1}}{ds^m}\left[\frac{(s-s_j)^m\varphi(s)}{\psi(s)}\right]e^{s_j\tau}.$$

Example. The mapping is

$$\bar{u}(s)=\frac{\operatorname{ch}\sqrt{\frac{s+b}{a}}(L-x)}{s(s+b)\operatorname{ch}\sqrt{\frac{s+b}{a}}L}\quad(a>0,\ b>0).$$

[1]The nulls of function f(s) refer to points at which f(s) = 0. Analytic function f(s) has at point $s = s_0$ a null of order m, where m is a positive integer, if its Taylor expansion in the vicinity of this point is

$$f(s)=\sum_{n=0}^{\infty}c_n(s-s_0)^n.$$

Find the original $u(\tau)$.

We know that

$$\operatorname{ch} z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$$

Therefore, in spite of the fact that the expression for $\bar{u}(s)$ contains $\sqrt{(s + b)/a}$, function $\bar{u}(s)$ is unambiguous. This function, furthermore, is meromorphic, it has simple poles s = 0 and s = -b, as well as a denumerable set of poles s_n, which are roots of the equation

$$\operatorname{ch} \sqrt{\frac{s+b}{a}} L = \cos i \sqrt{\frac{s+b}{a}} L = 0.$$

The roots of this equation are

$$s_n = -\frac{(2n-1)^2 \pi^2 a}{4L^2} - b = -\frac{\mu_n^2 a}{L^2} - b \quad (n = 1, 2 \dots),$$

where

$$\mu_n = -\frac{(2n-1)\pi}{2}.$$

We note that all poles of function $\bar{u}(s)$ lie on the negative real half axis. In connection with this, we select as the integration path in the Riemann-Mellin inversion formula the line Re s = σ, parallel to the imaginary axis and lying to the right of it, adding the direct sequence of arcs C_{R_n} of circles with radius

$$R_n = \frac{n^2 \pi^2 a}{L^2} + b$$

and with centers at the coordinate origin. The arcs are located to the left of line Re s = σ (Figure 3-2).

It is not difficult to see that the arcs which we have constructed pass through no singular point of function $\overline{u}(s)$.

Let us show that on arcs C_{R_n} the function

$$\Phi(s) = \frac{\operatorname{ch} \sqrt{\frac{s+b}{a}}(L-x)}{\operatorname{ch} \sqrt{\frac{s+b}{a}} L}$$

is limited.

The modulus of function $\Phi(s)$ is

$$|\Phi(s)| = \left| \frac{\operatorname{ch} \sqrt{\frac{s+b}{a}}(L-x)}{\operatorname{ch} \sqrt{\frac{s+b}{a}} L} \right| =$$

$$= \left| \frac{\exp\left[\sqrt{\frac{s+b}{a}}(L-x)\right] - \exp\left[-\sqrt{\frac{s+b}{a}}(L-x)\right]}{\exp\left[\sqrt{\frac{s+b}{a}} L\right] + \exp\left[-\sqrt{\frac{s+b}{a}} L\right]} \right|.$$

Since for all points s on arc ABCDБ, in addition to point C on the real negative half axis

$$\operatorname{Re} s \sqrt{s} > 0^{1} \text{ и } |e^{-k\sqrt{s}}| = e^{-k \operatorname{Re} \sqrt{s}}, \ldots$$

then with sufficiently great R_n

$$|\Phi(s)| \to \frac{\left|\exp\left[\sqrt{\frac{s+b}{a}}(L-x)\right]\right|}{\left|\exp\left[\sqrt{\frac{s+b}{a}} L\right]\right|} = \left|\exp\left[-\sqrt{\frac{s+b}{a}} x\right]\right|$$

---

[1]For points on arcs AB and DБ this is obvious; for points on arc BCD it should be kept in mind that if s is located in the left half plane, $\sqrt{s}$ lies in the right half plane.

and, consequently, as $R_n \to \infty$ ($|s| \to \infty$)

$$\Phi(s) \to 0.$$

On the real axis, function $\Phi(s)$ is

$$\Phi(s) = (-1)^{n+1} \cos \frac{n\pi}{L}(L - x) \quad (n = 1, 2 \ldots),$$

i.e., remains limited with any n.

Function $\bar{u}(s)$ contains, in addition to $\Phi(s)$, the factor $1/s(s + b)$. From this it becomes obvious that on the system of arcs C_{R_n} the function satisfies the conditions of the Jordan lemma and is absolutely integrable on line $(\sigma - i\infty, \sigma + i\infty)$.

Thus, the second theorem of the expansion is applicable to mapping $\bar{u}(s)$ and

$$u(\tau) = \sum_{s_j} \operatorname{res} \bar{u}(s_j)\, e^{s_j \tau}.$$

Since in the vicinity of their simple poles $s = 0$, $s = -b$ and

$$s_n = -\frac{(2n-1)^2 \pi^2 a}{4L^2} - b \quad (n = 1, 2 \ldots)$$

function $\bar{u}(s)$ is represented as the ratio of two analytic functions

$$\bar{u}(s) = \frac{\varphi(s)}{\psi(s)},$$

where

$$\varphi(s) = \operatorname{ch} \sqrt{\frac{s+b}{a}}(L - x);$$

$$\psi(s) = s(s+b) \operatorname{ch} \sqrt{\frac{s+b}{a}} L,$$

then

$$u(\tau) = \sum_{s_j} \frac{\varphi(s_j)}{\psi'(s_j)} e^{s_j \tau}.$$

Derivative $\psi'(s)$ is equal to

$$\psi'(s) = s \operatorname{ch} \sqrt{\frac{s+b}{a}} L + (s+b) \operatorname{ch} \sqrt{\frac{s+b}{a}} L + \frac{1}{2\sqrt{a}} s \sqrt{s+b} \operatorname{sh} \sqrt{\frac{s+b}{a}} L.$$

From this

$$\psi'(s)_{s\to 0} = b \operatorname{ch} \sqrt{\frac{b}{a}} L; \quad \varphi(s)_{s\to 0} = \operatorname{ch} \sqrt{\frac{b}{a}} (L-x);$$

$$\psi'(s)_{s\to -b} = -b; \quad \varphi(s)_{s\to -b} = 1;$$

$$\psi'(s)_{s\to s_n} = \frac{L}{2\sqrt{a}} s_n \sqrt{s_n + b} \operatorname{sh} \sqrt{\frac{s_n + b}{a}} L = \frac{\mu_n}{2} \left(\frac{\mu_n^2 a}{L^2} + b \right) \sin \mu_n;$$

$$\varphi(s)_{s\to s_n} = \sin \mu_n \sin \mu_n \frac{x}{L}.$$

Thus

$$u(\tau) = \frac{1}{b} \left[\frac{\operatorname{ch} \sqrt{\frac{b}{a}} (L-x)}{\operatorname{ch} \sqrt{\frac{b}{a}} L} - e^{-b\tau} \right] +$$

$$+ \frac{L^2}{a} e^{-b\tau} \sum_{n=1}^{\infty} \frac{A_n \sin \mu_n \frac{x}{L}}{\mu_n^2 + b^{*2}} e^{-\mu_n^2 \frac{a\tau}{L^2}},$$

where

$$\mu_n = \frac{(2n-1)\pi}{2}; \quad A_n = \frac{2}{\mu_n}; \quad b^{*2} = \frac{bL^2}{a}.$$

Application of the Laplace Transform to Solve the Problem of Heat Conductivity

The Laplace transform is used to solve the problem of heat conductivity for unlimited, semilimited and finite bodies with heterogeneous differential equations and heterogeneous, sometimes complex, boundary conditions. The methodology of solution of problems of widely varied nature, as we will see below, is the same; here, as a rule, no difficulties arise in relation to the convergence of the series. The Laplace transform can be used to produce solutions convenient for numerical realization with both long and short times.

The presence of a large number of detailed tables of direct and inverse transforms, of simple theorems, the possibility of using the powerful apparatus of the theory of functions of a complex variable, have all facilitated extensive application of the method of the Laplace transform in the analytic theory of heat conductivity.

The Laplace transform cannot be applied to equations for which the coefficients before the temperature function or its derivatives depend on time. Certain difficulties also arise in those cases when the initial conditions are functions of the coordinates.

1. One-dimensional problems. Let us analyze the solution to the following problem:

$$\frac{\partial T}{\partial \tau}=a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right)+p(\xi)T+\frac{1}{c\gamma}q(\xi,\tau)\;(R_1<\xi<R_2,\ \tau>0,\ i=0\vee 1); \tag{3-115}$$

$$T(\xi,0)=f(\xi)\;(R_1\leqslant\xi\leqslant R_2); \tag{3-116}$$

$$\alpha_j\frac{\partial T(R_j,\tau)}{\partial \xi}+(-1)^j\beta_j T(R_j,\tau)=\gamma_j g_j(\tau)\;(j=1,2). \tag{3-117}$$

Suppose the fixed functions $q(\xi, \tau)$ and $g_j(\tau)$ satisfy all conditions assuring the existence of a mapping.

We can then write

$$\bar{g}(\xi,s)=\int_0^\infty q(\xi,\tau)\,e^{-s\tau}\,d\tau;$$

$$\bar{g}_j(s)=\int_0^\infty g(\tau)\,e^{-s\tau}\,d\tau.$$

Let us assume further that the desired temperature function $T(\xi, \tau)$ is a piecewise-smooth function of time where $\tau \geq 0$, equal to 0 where $\tau < 0$ and having growth exponent b when $\tau > 0$, so that

$$\bar{T}(\xi, s) = \int_0^\infty T(\xi, \tau) e^{-s\tau} d\tau; \tag{3-18}$$

$$T(\xi, \tau) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \bar{T}(\xi, s) e^{s\tau} ds.\text{[1]} \tag{3-19}$$

Let us multiply equation (3-115) and boundary conditions (3-117) by $e^{-s\tau}$ and integrate from 0 to ∞. We then have:

$$\int_0^\infty \frac{\partial T}{\partial \tau} e^{-s\tau} d\tau = a \int_0^\infty \frac{1}{\xi^i} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) e^{-s\tau} d\tau + \frac{1}{c\gamma} \int_0^\infty q(\xi, \tau) e^{-s\tau} d\tau;$$

$$\alpha_j \int_0^\infty e^{-s\tau} \lim_{\xi \to R_j} \frac{\partial T(\xi, \tau)}{\partial \xi} d\tau + (-1)^j \beta_j \int_0^\infty e^{-s\tau} \lim_{\xi \to R_j} T(\xi, \tau) d\tau =$$

$$= \gamma_j \int_0^\infty e^{-s\tau} g_j(\tau) d\tau \quad (j = 1, 2)\text{[2]}.$$

The mapping of the derivative with respect to time is:

$$\int_0^\infty \frac{\partial T}{\partial \tau} e^{-s\tau} d\tau = s\bar{T}(\xi, s) - T(\xi, 0) = s\bar{T} - f(\xi).$$

---

[1]In formula (3-119) in contrast to equation (3-115), i is an imaginary unit.

[2]We recall that in the general case, the boundary conditions should be taken as the limiting relationships

$$\alpha_1 \lim_{\xi \to R_1 + 0} \frac{\partial T(\xi, \tau)}{\partial \xi} - \beta_1 \lim_{\xi \to R_1 + 0} T(\xi, \tau) = \gamma_1 g_1(\tau);$$

$$\alpha_2 \lim_{\xi \to R_2 - 0} \frac{\partial T(\xi, \tau)}{\partial \xi} + \beta_2 \lim_{\xi \to R_2 - 0} T(\xi, \tau) = \gamma_2 g_2(\tau).$$

Let us assume that we can interchange integration with respect to time and differentiation with respect to the coordinates, as well as integration and a passage to the limit.

Then

$$\int_0^\infty \frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial T}{\partial\xi}\right)e^{-s\tau}d\tau=\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial}{\partial\xi}\int_0^\infty Te^{-s\tau}d\tau\right)=\frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i\frac{d\bar{T}}{d\xi}\right);$$

$$\int_0^\infty e^{-s\tau}\lim_{\xi\to R_j}\frac{\partial T(\xi,\tau)}{\partial\xi}d\tau=\lim_{\xi\to R_j}\frac{\partial}{\partial\xi}\int_0^\infty Te^{-s\tau}d\tau=\frac{d\bar{T}(R_j,s)}{d\xi};$$

$$\int_0^\infty e^{-s\tau}\lim_{\xi\to R_j}T(\xi,\tau)\,d\tau=\lim_{\xi\to R_j}\int_0^\infty Te^{-s\tau}d\tau=\bar{T}(R_j,s)$$

and the transformed (or supplementary) equation and boundary conditions become:

$$\frac{1}{\xi^i}\frac{d}{d\xi}\left(\xi^i\frac{d\bar{T}}{d\xi}\right)-\frac{1}{a}(s-p(\xi))\bar{T}+\frac{1}{\lambda}\bar{q}(\xi,s)-\frac{1}{a}f(\xi)=0; \tag{3-120}$$

$$\alpha_j\frac{d\bar{T}(\xi,s)}{d\xi}+(-1)^j\beta_j\bar{T}(\xi,s)=\gamma_j\bar{g}_j(s)\quad(j=1,2). \tag{3-121}$$

The supplementary equation (3-120) is an ordinary second order differential equation. Its solution satisfying boundary conditions (3-121) is the mapping $\bar{T}(\xi, s)$.

Inverse transformation of function $\bar{T}(\xi, s)$ by one of the methods indicated earlier yields the desired temperature function $T(\xi, \tau)$.

2. Two-dimensional and three-dimensional problems. Without reducing generality, let us discuss the solution of the following two-dimensional problem:

$$\frac{\partial T}{\partial\tau}=a\left[\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial T}{\partial\xi}\right)+\frac{\partial^2T}{\partial\zeta^2}\right]+p(\zeta)T+Q(\xi,\zeta,\tau) \tag{3-122}$$

$$(R_1<\xi<R_2,\ L_1<\zeta<L_2,\ \tau>0,\ i=0\vee 1);$$

$$T(\xi,\zeta,0)=f(\xi,\zeta)\quad(R_1\leqslant\xi\leqslant R_2,\ L_1\leqslant\zeta\leqslant L_2); \tag{3-123}$$

$$\alpha_j\frac{\partial T(R_j,\zeta,\tau)}{\partial\xi}+(-1)^j\beta_jT(R_j,\zeta,\tau)=\gamma_jg_j(\zeta,\tau)\quad(j=1,2); \tag{3-124}$$

$$\alpha_k\frac{\partial T(\xi,L_{k-2},\tau)}{\partial\zeta}+(-1)^k\beta_kT(\xi,L_{k-2},\tau)=\gamma_kg_k(\xi,\tau)\quad(k=3,4).$$

The direct application of the integral Laplace transform to differential equation (3-122) and boundary conditions (3-124) yields

$$a\left[\frac{1}{\xi^i}\frac{\partial}{\partial\xi}\left(\xi^i\frac{\partial \bar{T}}{\partial\xi}\right)+\frac{\partial^2\bar{T}}{\partial\zeta^2}\right]-(s-p(\zeta))\bar{T}+f(\xi,\zeta)+$$
$$+\bar{Q}(\xi,\zeta,s)=0\ (R_1<\xi<R_2,\ L_1<\zeta<L_2); \qquad (3\text{-}125)$$
$$\alpha_j\frac{\partial\bar{T}(R_j,\zeta,s)}{\partial\xi}+(-1)^j\beta_j\bar{T}(R_j,\zeta,s)=\gamma_j\bar{g}_i(\zeta,s)\quad (j=1,2);$$
$$\alpha_k\frac{\partial\bar{T}(\xi,L_{k-2},s)}{\partial\zeta}+(-1)^k\beta_k\bar{T}(\xi,L_{k-2},s)=\gamma_k g_k(\xi,s)\ (k=3,4). \qquad (3\text{-}126)$$

Supplementary equation (3-125) is an equation in partial derivatives of elliptical type, containing one variable less than the initial equation (3-122). To integrate equation (3-125), we can use the methods presented in the previous paragraphs.

One effective means of solving problem (3-122)-(3-124) is combined application of the integral transforms with respect to spatial variables and the Laplace transform with respect to time. Since in equation (3-122) $p = p(\zeta)$, the integral transform with respect to the spatial variable can only by the transform with respect to ξ.

Let us assume for definition that the area of change of ξ is the finite interval $[R_1, R_2]$, at the ends of which the boundary conditions of the third kind below are fixed

$$\frac{\partial T(R_j,\zeta,\tau)}{\partial\xi}=(-1)^j h_j[\psi_j(\zeta,\tau)-T(R_j,\zeta,\tau)]\quad (j=1,2).$$

After performing a finite integral transform, defined by the formula

$$\tilde{T}_n=\int_{R_1}^{R_2}\xi^i T U_0\left(\mu_n\frac{\xi}{R}\right)d\xi, \qquad (3\text{-}127)$$

where $U_0(\mu_n\frac{\xi}{R})$ is the Eigenfunction; μ_n is the root of the characteristic equation; R is the characteristic dimension of the body in direction ξ, we produce:

$$\frac{\partial T_n}{\partial \tau} = a\frac{\partial^2 \widetilde{T}_n}{\partial \zeta^2} - \frac{a\mu_n^2}{R^2}\widetilde{T}_n + p(\zeta)\widetilde{T}_n + \widetilde{Q}_n(\zeta, \tau) +$$

$$+ R_2^i h_2 \psi_2(\zeta, \tau) U_0\left(\mu_n \frac{R_2}{R}\right) + R_1^i h_1 a \psi_1(\zeta, \tau) U_0\left(\mu_n \frac{R_1}{R}\right) \tag{3-128}$$

$$(L_1 < \zeta < L_2,\ \tau > 0,\ i = 0 \vee 1);$$

$$\widetilde{T}_n(\zeta, 0) = \widetilde{f}_n(\zeta)\ (L_1 \leq \zeta \leq L_2); \tag{3-129}$$

$$a_k \frac{\partial T_n(L_{k-2}, \tau)}{\partial \zeta} + (-1)^k \beta_k \widetilde{T}_n(L_{k-2}, \tau) = \gamma_k \widetilde{g}_{k(n)}(\tau)\ (k=3, 4). \tag{3-130}$$

Let us now subject equation (3-127) and boundary condition (3-129) to a Laplace transform with respect to time.

We have the supplementary equation

$$a\frac{d^2\bar{T}_n}{d\zeta^2} - \left(s + \frac{a\mu_n^2}{R^2} - p(\zeta)\right)\bar{\widetilde{T}}_n + [\widetilde{f}_n(\zeta) + \bar{\widetilde{Q}}_n(\zeta, s) +$$

$$+ R_2^i h_2 \bar{\psi}_2(\zeta, s) U_0\left(\mu_n \frac{R_2}{R}\right) + R_1^i h_1 a \bar{\psi}_1(\zeta, s) U_0\left(\mu_n \frac{R_1}{R}\right) = 0; \tag{3-131}$$

the boundary conditions

$$a_k \frac{d\bar{\widetilde{T}}_n(L_{k-2}, s)}{d\zeta} + (-1)^k \beta_k \bar{\widetilde{T}}_n(L_{k-2}, s) = \gamma_k \bar{\widetilde{g}}_k(s)\ (k=3, 4). \tag{3-132}$$

Here

$$\bar{\widetilde{T}}_n = \int_0^\infty \widetilde{T} e^{-s\tau} d\tau.$$

The solution of the ordinary second order differential equation (3-131), satisfying the boundary conditions (3-132), is the twice transformed temperature $\bar{\widetilde{T}}_n(\zeta, s)$.

The desired temperature function $T(\xi, \zeta, \tau)$ is determined as a result of the inverse transform of function $\bar{\widetilde{T}}_n(\zeta, s)$ first using a Laplace transform, then using the inversion formula

$$T(\xi, \zeta, \tau) = \sum_{n=1}^{\infty} \frac{\overline{T}_n(\zeta, \tau)}{\| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right),$$

where

$$\| U_0 \|^2 = \int_{R_1}^{R_2} \xi^i U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi,$$

corresponding to the finite integral transform (3-127).

In passing we note that in most cases the sequence of transforms, both direct and inverse, is insignificant.

Note. If $p = p(\xi)$, we should apply a transform with respect to ζ. Where $p = p_0$ = const, multiple transforms with respect to both coordinates ξ and ζ are possible.

Example. Establish the temperature field of a half space, the initial temperature of which is 0. On the surface of the half space we assign boundary conditions of the third kind, the ambient temperature is T_c = const.

This is a one-dimensional problem. We place the coordinate origin on the surface and direct the x axis into the depth of the mass.

The problem is then formulated as follows.

The differential equation

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2} \quad (0 < x < \infty, \ \tau > 0). \qquad (3\text{-}133)$$

Initial condition

$$T(x, 0) = 0 \quad (0 \leqslant x \leqslant \infty). \qquad (3\text{-}134)$$

Boundary conditions

$$\frac{\partial T(0, \tau)}{\partial x} = -h\,[T_c - T(0, \tau)]; \qquad (3\text{-}135)$$

$T(\infty, \tau) \neq \infty$ (conditionof finiteness of temperature function).

Applying the Laplace transform, we produce the supplementary equation

$$\frac{d^2\bar{T}}{dx^2}-\frac{s}{a}\bar{T}=0$$

and the boundary conditions

$$\frac{d\bar{T}(0,s)}{dx}=-h\left[\frac{\bar{T}_c}{s}-\bar{T}(0,s)\right];$$

$$\bar{T}(\infty,s)\neq\infty.$$

The solution of the supplementary equation is

$$\bar{T}=\frac{h\sqrt{a}\,T_c}{s(h\sqrt{a}+\sqrt{s})}\,e^{-\frac{x}{\sqrt{a}}\sqrt{s}}.$$

From Laplace transformation table [69] we find

$$\mathcal{L}^{-1}\left[\frac{e^{-k\sqrt{s}}}{s(b+\sqrt{s})}\right]=\frac{1}{b}\left[\operatorname{erfc}\frac{k}{2\sqrt{\tau}}-e^{bk}e^{b^2\tau}\operatorname{erfc}\left(b\sqrt{\tau}+\frac{k}{2\sqrt{\tau}}\right)\right],$$

where

$$\operatorname{erfc} z=1-\operatorname{erf} z,$$

$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}\,dz$ is the Gaussian error function [69].

Thus,

$$T=T_c\left[\operatorname{erfc}\frac{x}{2\sqrt{a\tau}}-e^{hx+h^2a\tau}\operatorname{erfc}\left(h\sqrt{a\tau}+\frac{x}{2\sqrt{a\tau}}\right)\right].$$

3-5. The Method of the Green Function

The Dirac Delta Function

The Dirac delta function $\delta(\xi)$ refers to a function equal to 0 with all $\xi > 0$ and $\xi < 0$ and infinity where $\xi = 0$, for which the integral

$$\int_{-\infty}^{\infty} \delta(\xi)\, d\xi = 1.^{1} \qquad (3\text{-}136)$$

These properties of the delta function lead to the basic relationship

$$\int_{-\infty}^{\infty} f(\xi)\,\delta(\xi)\, d\xi = f(0). \qquad (3\text{-}137)$$

Sometimes formula (3-137) is taken as a definition of the delta function.

Let us note a number of other properties of the delta function:

$$\begin{gathered}
\int_{-\infty}^{\infty} f(\xi)\,\delta(\xi - a)\, d\xi = f(a); \\
f(\xi)\,\delta(\xi - a) = f(a)\,\delta(\xi - a)^{2}; \\
\delta(a\xi) = \frac{1}{|a|}\,\delta(\xi) \quad (a = \mathrm{const} \neq 0); \\
\delta(-\xi) = \delta(\xi); \\
\int_{-\infty}^{\infty} f(\xi)\,\delta^{(n)}(\xi - a)\, d\xi = (-1)^{n} f^{(n)}(a).
\end{gathered} \qquad (3\text{-}138)$$

The delta function also has a sense in the plane $-\delta(x, y)$, in the space $-\delta(x, y, z)$ and in time $-\delta(\tau)$, where

$$\delta(x, y, z, \tau) = \delta(x)\,\delta(y)\,\delta(z)\,\delta(\tau). \qquad (3\text{-}139)$$

The integral of the delta function of the following form

$$e(\xi) = \int_{-\infty}^{\infty} \delta(\xi)\, d\xi \qquad (3\text{-}140)$$

---

[1] A more precise definition of the delta function is given in special guides on generalized functions [24].

[2] The properties of the delta function in (3-137) and (3-138) are frequently called "capturing" or "filtering."

is called the unit function $e(\xi)$.

It follows from definition (3-136) that unit function $e(\xi)$ is a function which is discontinuous where e = 0, equal to

$$e(\xi)=\begin{cases} 1 & \text{where } \xi>0; \\ 0 & \text{where } \xi<0. \end{cases}$$

Similarly

$$e(\xi-a)=\begin{cases} 1 & \text{where } \xi-a>0; \\ 0 & \text{where } \xi-a<0. \end{cases}$$

The derivative of the unit function is the delta function

$$\frac{de(\xi)}{d\xi}=\delta(\xi); \qquad \frac{de(\xi-a)}{d\xi}=\delta(\xi-a). \tag{3-141}$$

The integral of the unit function

$$\int_{-\infty}^{\xi} e(\xi)\,d\xi=\begin{cases} 0 & \text{where } \xi<0; \\ \xi & \text{where } \xi>0 \end{cases}$$

is a continuous function.

Fundamental Solutions of the Differential Equation for Heat Conductivity

The differential equation for heat conductivity for an unlimited body

$$\frac{\partial T}{\partial\tau}=a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

$$(-\infty<x<\infty,\ -\infty<y<\infty,\ -\infty<z<\infty,\ \tau>0)$$

satisfies the function

$$T(x, y, z, \tau)=\frac{b_1}{(2\sqrt{\pi a\tau})^3}\exp\left[-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a\tau}\right]. \tag{3-142}$$

It is not difficult to see that

$$\lim_{\substack{\tau\to 0,\\ x\neq x_0,\\ y\neq y_0,\\ z\neq z_0}} T(x, y, z, \tau) = 0, \qquad \lim_{\substack{\tau\to 0,\\ x=x_0,\\ y=y_0,\\ z=z_0,}} T(x, y, z, \tau) = \infty,$$

i.e., temperature function (3-142) as $\tau \to 0$ everywhere except at points (x_0, y_0, z_0), where it becomes infinite.

The full quantity of heat in a body, resulting from temperature function (3-142),

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\gamma T(x, y, z, \tau)\,dx\,dy\,dz = b_1 c\gamma \left\{\frac{1}{2\sqrt{\pi a\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4a\tau}}dx\right\}\times$$
$$\times\left\{\frac{1}{2\sqrt{\pi a\tau}}\int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{4a\tau}}dy\right\}\left\{\frac{1}{2\sqrt{\pi a\tau}}\int_{-\infty}^{\infty} e^{-\frac{(z-z_0)^2}{4a\tau}}dz\right\}^{1} = b_1 c\gamma$$

is independent of time, though it can be shown that for sufficiently small intervals of time in the vicinity of $\tau = 0$, an arbitrarily large portion of the heat is enclosed in a sphere of arbitrarily small radius, surrounding point (x_0, y_0, z_0).

This means that heat has been introduced into the body as a result of instantaneous liberation at $\tau = 0$ at point (x_0, y_0, z_0).

This source of heat is called *an instantaneous point source*, the quantity b_1 is called its power. Consequently, function (3-142) describes the temperature field in an unlimited body, arising due to the action at point (x_0, y_0, z_0) at point in time $\tau = 0$ of an instantaneous point heat source of constant power b_1.

---

[1] The substitutions $(x - x_0)/2\sqrt{a\tau} = \xi$, etc. convert the expressions in the braces to the known integrals

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\xi^2}d\xi = 1.$$

If we integrate expression (3-142) with respect to z from -∞ to ∞, we arrive at a temperature function of the form

$$T(x, y, \tau) = \frac{b_2}{4\pi a\tau} \exp\left[-\frac{(x-x_0)^2+(y-y_0)^2}{4a\tau}\right], \tag{3-143}$$

which is a solution to the two-dimensional differential equation of heat conductivity

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) (-\infty < x < \infty, \ -\infty < y < \infty, \ \tau > 0).$$

Function (3-143) describes the temperature field in an unlimited body, arising due to the application at moment in time $\tau = 0$ of instantaneous point heat sources of constant power, evenly distributed along line $x = x_0$, $y = y_0$, parallel to the z axis. This source is called an instantaneous linear source, b_2 is its power.

Integration of expression (3-143) with respect to y from -∞ to ∞ leads to the temperature function

$$T(x, \tau) = \frac{b_3}{2\sqrt{\pi a\tau}} \exp\left[-\frac{(x-x_0)^2}{4a\tau}\right], \tag{3-144}$$

which is a solution to the one-dimensional differential equation of heat conductivity

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} (-\infty < x < \infty, \ \tau > 0).$$

Function (3-144) describes the temperature field in an unlimited body arising due to the application at moment in time $\tau = 0$ of instantaneous point sources of constant power, continually distributed over the plane $x = x_0$. In this case, we are dealing with an instantaneous planar heat source of power b_3.

The instantaneous point sources of heat of constant power, continuously distributed over a cylindrical surface $r = R_0$ forming an instantaneous cylindrical source cause a temperature field in an unlimited body which is described by the function

$$T(r, \tau) = \frac{b_2}{4\pi a\tau} \exp\left[-\frac{r^2 + R_0^2}{4a\tau}\right] I_0\left(\frac{rR_0}{2a\tau}\right). \tag{3-145}$$

Temperature function (3-145) is the solution to the differential equation of heat conductivity

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) (0 < r < \infty,\ \tau > 0).$$

The solution of the differential equation of heat conductivity provided by functions (3-142), (3-143), (3-144) and (3-145) where $b_1 = b_2 = b_3 = 1$ are called the fundamental or source-type solutions.

The Green Function of the Heat Conductivity Problem

The Green function (influence function) of the heat conductivity problem refers to a function defining the temperature at point $\mathcal{T}$ of a body resulting from the action at moment in time $\tau = t$ at point $\mathcal{T}_0$ of an instantaneous heat source of unit power[1]; the initial temperature of the body and the conditions of heat exchange on its surface are homogeneous.

The Green function will be represented by G.

The Green function depends on the difference $\tau - t$ and the position of point $\mathcal{T}$ and $\mathcal{T}_0$, relative to which it is symmetrical, i.e.,

$$G = G(\mathcal{P}, \mathcal{P}_0, \tau - t) = G(\mathcal{P}_0, \mathcal{P}, \tau - t).$$

It follows from the definition presented above that

$$G(\mathcal{P}, \mathcal{P}_0, \tau - t) = 0 \text{ where } \tau < t.$$

In accordance with the number of spatial variables, we distinguish three-dimensional, two-dimensional and one-dimensional Green functions.

The three-dimensional Green function $G(x, y, z, x_0, y_0, z_0, \tau - t)$ is generated by an individual instantaneous point heat source, located at point (x_0, y_0, z_0).

[1]Instantaneous heat sources of unit power will subsequently be called unit heat sources.

The two-dimensional Green function $G(\xi, \zeta, \xi_0, \zeta_0, \tau - t)$ is generated: for a heat conductivity problem in rectangular coordinates $\xi = x$, $\zeta = y$ -- by a unit instantaneous linear heat source, distributed over line $x = x_0$, $y = y_0$, parallel to the z axis; for a problem in heat conductivity in cylindrical coordinates $\xi = r$, $\zeta = z$ -- by a unit instantaneous cylindrical heat source distributed over the surface $r = R_0$ of finite length L_0.

The one-dimensional function $G(\xi, \xi_0, \tau - t)$ is generated: for the problem of heat conductivity in rectangular coordinates $\xi = x$ -- by a unit instantaneous planar heat source, distributed over the surface $x = x_0$; for the problem of heat conductivity in cylindrical coordinates $\xi = r$ -- by a unit instantaneous cylindrical heat source, distributed over the cylindrical surface $r = R_0$.

The Green function occupies a singular position in the theory of heat conductivity. This results from the fact that the Green function makes it possible to describe the solution of the problem of heat conductivity for a fixed area with fixed edge conditions in the form of quadratures. Let us explain this using specific examples.

The solution of the three-dimensional heterogeneous differential equation

$$\frac{\partial T(x, y, z, \tau)}{\partial \tau} = a\nabla^2 T(x, y, z, \tau) + p(x, y, z, \tau) T(x, y, z, \tau) + Q(x, y, z, \tau) \quad \text{(in the spatial area D, } \tau > 0) \qquad (3\text{-}146)$$

with heterogeneous initial condition

$$T(x, y, z, 0) = f(x, y, z) \quad \text{(in closed area D, } \tau = 0) \qquad (3\text{-}147)$$

and the heterogeneous boundary conditions of general form

$$\alpha \frac{\partial T}{\partial n} + \beta T = \gamma g(x, y, z, \tau) \quad \text{(on surface S, limiting area D, } \tau > 0) \qquad (3\text{-}148)$$

where n is a normal to surface S, α, β, γ are parameters[1], is written as follows:

[1]Boundary conditions written in this manner can be made specific with boundary conditions of the first, second and third kinds. Parameters α, β and γ can generally be functions of coordinates and time.

$$T(x, y, z, \tau) = \iiint\limits_D (G)_{t=0} f(x_0, y_0, z_0)\, dV_0 +$$

$$+ a \int\limits_0^{\tau} \left[\iint\limits_S \left(G \frac{\partial T}{\partial n_0} - T \frac{\partial G}{\partial n_0} \right) dS_0 \right] dt + \int\limits_0^{\tau} \left[\iiint\limits_D GQ\, dV_0 \right] dt, \tag{3-149}$$

where $\partial/\partial n_0$ is the differentiation operator with respect to the coordinates x_0, y_0, z_0 in the direction of the external normal to the surface S; dV_0 and dS_0 are elements of the volume and surface in coordinates x_0, y_0, z_0.

For definition we assume that on surface S_1 we have assigned a boundary condition of the first kind ($\alpha = \alpha_1 = 0$, $\beta = \beta_1 = 1$, $\gamma = \gamma_1$, $g = g_1$), on surface S_2 -- the boundary condition of the second kind ($\alpha = \alpha_2$, $\beta = \beta_2 = 0$, $\gamma = \gamma_2$, $g = g_2$), on surface S_3 -- boundary condition of the third kind ($\alpha = \alpha_3$, $\beta = \beta_3$, $\gamma = \gamma_3$, $g = g_3$). Surfaces S_j ($j = 1, 2, 3$) are parts of surface S, so that $S_1 + S_2 + S_3 = S$. Under these conditions, the solution of problem (3-146)-(3-148) becomes:

$$T(x, y, z, \tau) = \iiint\limits_D (G)_{t=0} f(x_0, y_0, z_0)\, dV_0 -$$

$$- a \int\limits_0^{\tau} \left[\iint\limits_{S_1} \gamma_1 g_1 \frac{\partial G}{\partial n_0} dS_0 \right] dt + a \int\limits_0^{\tau} \left[\iint\limits_{S_2} \gamma_2 g_2 G\, dS_0 \right] dt -$$

$$- a \int\limits_0^{\tau} \left[\iint\limits_{S_3} \frac{\gamma_3}{\beta_3} g_3 \frac{\partial G}{\partial n_0} dS_0 \right] dt + \int\limits_0^{\tau} \left[\iiint\limits_D GQ\, dV_0 \right] dt. \tag{3-150}$$

The solution of the two-dimensional problem of heat conductivity is represented as

$$T(\xi, \zeta, \tau) = \iint\limits_B (G)_{t=0} f(\xi_0, \zeta_0)\, dB_0 + a \int\limits_0^{\tau} \left[\int\limits_{\Gamma} \left(G \frac{\partial T}{\partial n_0} - \right.\right.$$

$$\left.\left. - T \frac{\partial G}{\partial n_0} \right) d\Gamma \right] dt + \int\limits_0^{\tau} \left[\iint\limits_B GQ\, dB_0 \right] dt. \tag{3-151}$$

Here B is the calculation area; Γ is its boundary. In this case in rectangular coordinates: B is a planar area; Γ is the boundary curve; $\xi = x$, $\zeta = y$; $dB_0 = dx_0 dy_0$, $d\Gamma = dx_0$ or dy_0; in cylindrical coordinates with axial symmetry: B is the three-dimensional area; Γ is the boundary surface; $\xi = r$, $\zeta = z$; $dB_0 = 2\pi r_0 dr_0 dz_0$; $d\Gamma_0 = 2\pi R dz_0$ (on the cylindrical surface of radius R) or $d\Gamma_0 = 2\pi r_0 dr_0$ (on the flat end surface).

For one-dimensional problems in heat conductivity, we have

$$T(\xi, \tau) = \int_L (G)_{t=0} f(\xi_0)\, dL_0 + a \int_0^\tau \left[G \frac{\partial T}{\partial n_0} - T \frac{\partial G}{\partial n_0} \right]_\Gamma dt + \int_0^\tau \left[\int_L GQ\, dL_0 \right] dt. \tag{3-152}$$

Here L is the area of calculation; Γ is the boundary. In rectangular coordinates: L is the interval (R_1, R_2); Γ is its boundaries R_1 and R_2; $\xi = x_0$, $dL_0 = dx_0$; in cylindrical coordinates: L is the flat area, the boundary of which is defined by surfaces $r = R_1$ and $r = R_2$; $\xi = r_0$, $dL_0 = 2\pi r_0 dr_0$.

The formulas presented above, (3-150)-(3-152), are correct for all problems studied in this book.

Thus, the solution of these problems can be considered formally known if we know the corresponding Green functions.

Construction of the Green Function

First of all, let us discuss unidimensional Green functions. We will limit ourselves in this discussion to a presentation of the methods of construction of Green functions by means of the methods of integral transforms.

As was indicated earlier, a one-dimensional Green function in rectangular coordinates describes the temperature field which arises in time $\tau > t$ following an action at moment in time $\tau = t$ applied by an individual instantaneous heat source placed at point $x = x_0$; the initial and boundary conditions for the Green function are homogeneous. It follows from this that by means of delta functions, the problem of seeking out function $G(x, x_0, \tau - t)$ can be formulated as

$$\frac{\partial G}{\partial \tau} = a\frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\,\delta(\tau - t)^{[1]} \; (R_1 < x,\ x_0 < R_2,\ \tau > 0,\ 0 < t < \tau); \tag{3-153}$$

$$G(x,\ x_0,\ \tau = 0) = 0 \; (R_1 \leq x,\ x_0 \leq R_2); \tag{3-154}$$

$$\alpha_j\frac{\partial G}{\partial n} + \beta_j G = 0 \; (j = 1,\ 2) \tag{3-155}$$
(at the boundary of the interval, $\tau > 0$).

Suppose for definition we analyze the problem of heat conductivity for half space ($0 \leq x,\ x_0 < \infty$) with boundary conditions of the first kind. In this case, boundary condition (3-155) becomes specific

$$G|_{x=0} = 0.$$

The Green function can be found by means of a Laplace transform. We will base ourselves on the solution for an individual instantaneous planar heat source acting at moment $\tau = 0$ on plane $x = x_0$ of the unlimited medium

$$u = \frac{1}{2\sqrt{\pi a\tau}}\, e^{-\frac{(x-x_0)^2}{4a\tau}}.^{[2]}$$

Let us assume:

$$G|_{x=0} = u + v. \tag{3-156}$$

Green function G and the source function u satisfy the differential equation

$$\frac{\partial W}{\partial \tau} = a\frac{\partial^2 W}{\partial x^2} \; (W = G,\ u).$$

This same equation should also satisfy function v, i.e.

---

[1] In equation (3-153), we assume $(x, \tau) = 0$. Subsequently, we will extend the results produced to $p(x, \tau) \neq 0$.

[2] As we can easily see, function $u(x, x_0, \tau)$ is a fundamental solution of the heat conductivity equation.

$$\frac{\partial v}{\partial \tau} = a\frac{\partial^2 v}{\partial x^2}. \qquad (3\text{-}157)$$

The edge conditions for function v will be:

$$v|_{\tau=0} = 0; \qquad (3\text{-}158)$$

$$v|_{x=0} = -u|_{x=0}{}^{1}, \; v|_{x\to\infty} \neq \infty. \qquad (3\text{-}159)$$

Performing Laplace transforms on relationship (3-156), equation (3-157) and boundary conditions (3-159), as defined by the formula

$$\overline{W} = \int_0^\infty W e^{-s\tau} d\tau,$$

we produce

$$\begin{gathered} \overline{G} = \overline{u} + \overline{v}; \\ \frac{d^2\overline{v}}{dx^2} - \frac{s}{a}\overline{v} = 0; \\ \overline{v}|_{x=0} = -\overline{u}|_{x=0}; \; \overline{v}|_{x\to\infty} \neq \infty. \end{gathered} \qquad (3\text{-}160)$$

The solution of the supplementary problem (3-160) is:

$$\overline{v} = Ae^{-\sqrt{\frac{s}{a}}x} = -\overline{u}|_{x=0} e^{-\sqrt{\frac{s}{a}}x}.$$

From the direct Laplace transform tables, we know:

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi\tau}} e^{-\frac{k^2}{4\tau}}\right] = \frac{1}{\sqrt{s}} e^{-k\sqrt{s}}, \; k \geq 0.$$

---

[1]This condition follows from the boundary condition for the Green function

$$G|_{x=0} = u|_{x=0} + v|_{x=0} = 0.$$

Therefore

$$\bar{u} = \frac{1}{2\sqrt{as}} e^{-\sqrt{\frac{s}{a}}|x-x_0|}$$

and

$$\bar{v} = -\frac{1}{2\sqrt{as}} e^{-\sqrt{\frac{s}{a}}|x+x_0|}$$

Consequently

$$\bar{G}|_{t=0} = \bar{u} + \bar{v} = \frac{1}{2\sqrt{as}} \left(e^{-\sqrt{\frac{s}{a}}|x-x_0|} - e^{-\sqrt{\frac{s}{a}}|x+x_0|}\right).$$

From this, utilizing the tabular formula presented, we find:

$$G|_{t=0} = \frac{1}{2\sqrt{\pi a\tau}} \left[e^{-\frac{(x-x_0)^2}{4a\tau}} - e^{-\frac{(x+x_0)^2}{4a\tau}}\right].$$

Consequently, the desired Green function is:

$$G = \frac{1}{2\sqrt{\pi a(\tau - t)}} \left[e^{-\frac{(x-x_0)^2}{4a(\tau-t)}} - e^{-\frac{(x+x_0)^2}{4a(\tau-t)}}\right].$$

The problem of seeking out the Green function for a one-dimensional heat conductivity problem in a limited interval $[R_1, R_2]$ in rectangular and cylindrical coordinates is formulated as:

$$\frac{\partial G}{\partial \tau} = a\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial G}{\partial \xi}\right) + \frac{1}{(2\pi\xi)^i}\delta(\xi - \xi_0)\,\delta(\tau - t)$$

$$(R_1 < \xi < R_2,\ \tau > 0,\ i = 0 \vee 1);$$

$$G(\xi,\ \xi_0,\ \tau = 0) = 0\ (R_1 \leq \xi \leq R_2);$$

$$\alpha_j \frac{\partial G(R_j,\ \xi_0,\ \tau)}{\partial \xi}\Big|_\Gamma + \beta_j G(R_j,\ \xi_0,\ \tau)\Big|_\Gamma = 0\ (j = 1,\ 2).$$

Here $\xi = x$, $i = 0$ in rectangular coordinates; $\xi = r$, $i = 1$ in cylindrical coordinates.

Let us subject the differential equation and initial condition to a finite integral transform

$$\bar{G}_n = \int_{R_1}^{R_2} \xi^i G U_0\left(\mu_n \frac{\xi}{R}\right) d\xi; \tag{3-161}$$

$$G = \sum_{n=1}^{\infty} \frac{\bar{G}_n}{\| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right).^{1} \tag{3-162}$$

Here $U_0(\mu_n \frac{\xi}{R})$ is the Eigenfunction of the problem; $||U_0||^2 = \int_{R_1}^{R_2} \xi^i U_0^2(\mu_n \frac{\xi}{R}) d\xi$ is the square of the norm of the Eigenfunction; μ_n is the root of the characteristic equation of the problem; R is the characteristic size of the body.

We have:

$$\frac{d\bar{G}_n}{d\tau} = -\frac{\mu_n^2 a}{R^2} \bar{G}_n + \delta(\tau - t) \int_{R_1}^{R_2} \frac{1}{(2\pi\xi)^i} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) \delta(\xi - \xi_0)\, d\xi;$$

$$\bar{G}_n \big|_{\tau=0} = 0.$$

But

$$\int_{R_1}^{R_2} U_0\left(\mu_n \frac{\xi}{R}\right) \delta(\xi - \xi_0)\, d\xi = U_0\left(\mu_n \frac{\xi_0}{R}\right).$$

Therefore

[1]The case of boundary conditions of the second kind on both ends of the interval

$$\frac{\partial G}{\partial \xi}\bigg|_{\xi=R_1} = \frac{\partial G}{\partial \xi}\bigg|_{\xi=R_2} = 0 \quad \text{-- see below.}$$

$$\frac{d\bar{G}_n}{d\tau} = -\frac{\mu_n^2 a}{R^2}\bar{G}_n + \frac{1}{(2\pi)^i} U_0\left(\mu_n \frac{\xi_0}{R}\right)\delta(\tau - t);$$
$$\bar{G}_n|_{\tau=0} = 0.$$

From this transform $\bar{G}_n$ is equal to:

$$\bar{G}_n = e^{-\frac{\mu_n^2 a}{R^2}\tau} \frac{1}{(2\pi)^i} U_0\left(\mu_n \frac{\xi_0}{R}\right) \int_0^\tau e^{\frac{\mu_n^2 a}{R^2}\eta} \delta(\eta - t)\, d\eta$$

or

$$\bar{G}_n = \frac{1}{(2\pi)^i} e^{-\frac{\mu_n^2 a}{R^2}(\tau - t)} U_0\left(\mu_n \frac{\xi_0}{R}\right).$$

Based on the inversion formula (3-162), we find the Green function of the one-dimensional problem as

$$G(\xi, \xi_0, \tau - t) = \frac{1}{(2\pi)^i} \sum_{n=1}^{\infty} \frac{U_0\left(\mu_n \frac{\xi_0}{R}\right) U_0\left(\mu_n \frac{\xi}{R}\right)}{\|U_0\|^2} e^{-\frac{\mu_n^2 a}{R^2}(\tau - t)}.$$

Since with boundary conditions of the second kind, at both ends of the interval $[R_1, R_2]$ the characteristic equation of the problem contains the zero root $\mu_0 = 0$, in this case

$$G(\xi, \xi_0, \tau - t) = \frac{1}{(2\pi)^i} \lim_{\mu_n \to 0} \frac{U_0\left(\mu_n \frac{\xi_0}{R}\right) U_0\left(\mu_n \frac{\xi}{R}\right)}{\|U_0\|^2} +$$
$$+ \frac{1}{(2\pi)^i} \sum_{n=1}^{\infty} \frac{U_0\left(\mu_n \frac{\xi_0}{R}\right) U_0\left(\mu_n \frac{\xi}{R}\right)}{\|U_0\|^2} e^{-\frac{\mu_n^2 a}{R^2}(\tau - t)}.$$

The construction of the Green function of n-dimensional (two-dimensional and three-dimensional) heat conductivity problems is reduced to determination of the solution of the n-dimensional differential equation of heat conductivity

$$\frac{\partial G}{\partial \tau} = a\nabla^2 G + \delta(\tau - t) V(x_0, y_0, z_0),$$

satisfying the homogeneous initial condition

$$G|_{\tau=0} = 0 \quad .$$

and the homogeneous boundary conditions

$$\alpha \frac{\partial G}{\partial n} + \beta G = 0.$$

Here the function $V(x_0, y_0, z_0)$ is equal to:

in rectangular coordinates

$$V = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{(three-dimensional problem)}, \tag{3-163}$$

$$V = \delta(x - x_0)\delta(y - y_0) \quad \text{(two-dimensional problem)}; \tag{3-164}$$

in cylindrical coordinates

$$V = \frac{1}{2\pi r}\delta(r - r_0)\delta(z - z_0).$$

The Green function where $t = 0$, $G_{t=0} = G(x, u, z, x_0, y_0, z_0, \tau)$ can also be produced as a result of solution of the homogeneous heat conductivity equation

$$\frac{\partial G}{\partial \tau} = a\nabla^2 G$$

with the heterogeneous initial condition

$$G|_{\tau=0} = V(x_0, y_0, z_0)$$

and the boundary conditions

$$\alpha \frac{\partial G}{\partial n} + \beta G = 0.$$

As follows from formulas (3-163) and (3-164), the function $V(x_0, y_0, z_0)$ is equal to the product of the homogeneous delta functions. Consequently, according to the property of multiplication of solutions (see § 3-4), the n-dimensional Green function $G(x, y, z, x_0, y_0, z_0, \tau)$ and, therefore, $G(x, y, z, x_0, y_0, z_0, \tau - t)$ is equal to the product of the corresponding one-dimensional Green functions, i.e.

in rectangular coordinates

$$G(x, y, z, x_0, y_0, z_0, \tau - t) = G(x, x_0, \tau - t)\, G(y, y_0, \tau - t)\, G(z, z_0, \tau - t)$$

(three-dimensional problem),

$$G(x, y, x_0, y_0, \tau - t) = G(x, x_0, \tau - t)\, G(y, y_0, \tau - t)$$

(two-dimensional problem);

in cylindrical coordinates

$$G(r, z, r_0, z_0, \tau - t) = G(r, r_0, \tau - t)\, G(z, z_0, \tau - t).$$

Note. The construction of the Green function of the heat conductivity problem of the form

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right)\right] + p(\tau)T + Q(\xi, \tau)$$

$$(R_1 < \xi < R_2,\ \tau > 0,\ i = 0 \vee 1);$$

$$T(\xi, 0) = f(\xi)\ (R_1 \leqslant \xi \leqslant R_2);$$

$$\alpha_j \frac{\partial T(R_j, \tau)}{\partial \xi} + \beta_j T(R_j, \tau) = \gamma_j g_j(\tau)\ (j = 1, 2)$$

is reduced to determination of the solution of the differential equation

$$\frac{\partial G_1}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial G_1}{\partial \xi}\right)\right] + p(\tau) G_1 + \frac{1}{(2\pi\xi)^i}\delta(\tau - t)\delta(\xi - \xi_0),$$

satisfying the homogeneous edge conditions.

Let us assume

$$G_1 = G_0 \exp\left[\int_0^\tau p(\eta)\, d\eta\right].$$

Then

$$\frac{\partial G_0}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial G_0}{\partial \xi}\right)\right] + \frac{1}{(2\pi\xi)^i}\delta(\xi - \xi_0)\exp\left[-\int_0^t p(\eta)\, d\eta\right],^1$$

the edge conditions for the function G_0 will also be homogeneous. By the symbol G we represent the Green function satisfying the differential equation

$$\frac{\partial G}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial G}{\partial \xi}\right)\right] + \frac{1}{(2\pi\xi)^i}\delta(\tau - t)\delta(\xi - \xi_0)$$

and the homogeneous edge conditions.

---

[1]Since

$$\delta(\tau - t)\exp\left[-\int_0^\tau p(\eta)\, d\eta\right] = \delta(\tau - t)\exp\left[-\int_0^t p(\eta)\, d\eta\right].$$

Consequently,

$$G_1 = G \exp\left[\int_0^{\tau-t} p(\eta)\,d\eta\right].$$

This relationship allows us, on the basis of the Green function of the heat conductivity problem with differential equation

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T$$

to construct the Green function of the problem with the differential equation

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + p(\tau)\,T.$$

In conclusion, let us present a summary of one-dimensional Green functions of the heat conductivity problem with the differential equation

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right)\right] \quad (\xi = x,\ r;\ i = 0 \vee 1)$$

1. Unlimited body $(-\infty < x < \infty)$

$$G(x,\ x_0,\ \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}} \exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right]$$
$$(-\infty < x,\ x_0 < \infty,\ x \neq x_0,\ \tau > 0,\ 0 < t < \tau).$$

2. Half space $(0 \leq x < \infty)$:

a) boundary condition of first kind

$$G(x,\ x_0,\ \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right] - \exp\left[-\frac{(x + x_0)^2}{4a(\tau - t)}\right]\right\}$$
$$(0 < x,\ x_0 < \infty,\ x \neq x_0,\ \tau > 0,\ 0 < t < \tau);$$

b) boundary condition of second kind

$$G(x, x_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right] + \right.$$
$$\left. + \exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right]\right\};$$

c) boundary condition of third kind

$$G(x, x_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right] + \right.$$
$$\left. + \exp\left[-\frac{(x + x_0)^2}{4a(\tau - t)}\right] - 2h\int_0^\infty \exp\left[-\frac{(x + x_0 + \eta)^2}{4a(\tau - t)} - h\eta\right] d\eta\right\} =$$
$$= \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)}\right] + \exp\left[-\frac{(x + x_0)^2}{4a(\tau - t)}\right]\right\} -$$
$$- h\exp\left[h^2 a(\tau - t) + h(x + x_0)\right] \operatorname{erfc}\left[\frac{x + x_0}{2\sqrt{a(\tau - t)}} + h\sqrt{a(\tau - t)}\right].$$

3. An unlimited body ($0 \leq r < \infty$)

$$G(r, r_0, \tau - t) = \frac{1}{4\pi a(\tau - t)} \exp\left[-\frac{(r + r_0)^2}{4a(\tau - t)}\right] I_0\left(\frac{r r_0}{2a\sqrt{\tau - t}}\right)$$
$$(0 \leq r,\ r_0 < \infty,\ r \neq r_0,\ 0 < t < \tau).$$

4. An unlimited body with a circular notch ($R \leq r < \infty$):

a) boundary condition of the first kind

$$G(r, r_0, \tau - t) = \frac{1}{2\pi}\int_0^\infty e^{-\rho^2 a(\tau - t)} W(\rho r)\, W(\rho r_0)\, \rho\, d\rho$$
$$(R \leq r,\ r_0 < \infty,\ r \neq r_0,\ 0 < t < \tau),$$

where

$$W(\rho r) = \frac{Y_0(\rho R)\, J_0(\rho r) - J_0(\rho R)(Y_0 \rho r)}{[J_0^2(\rho R) + Y_0^2(\rho R)]^{1/2}};$$

b) boundary condition of the second kind

$$G(r, r_0, \tau - t) = \frac{1}{2\pi} \int_0^\infty e^{-\rho^2 a(\tau - t)} V(\rho r) V(\rho r_0) \rho \, d\rho$$

where

$$V(\rho r) = \frac{Y_1(\rho R) J_0(\rho r) - J_1(\rho R) Y_0(\rho r)}{[\rho J_1^2(\rho R) + \rho Y_1^2(\rho R)]^{1/2}};$$

c) boundary condition of the third kind

$$G(r, r_0, \tau - t) = \frac{1}{2\pi} \int_0^\infty e^{-\rho^2 a(\tau - t)} U(\rho r) U(\rho r_0) \rho \, d\rho,$$

where

$$U(\rho r) = \{[\rho Y_1(\rho R) + h Y_0(\rho R)] J_0(\rho r) - [\rho J_1(\rho R) + h J_0(\rho R)] Y_0(\rho r)\} \times$$
$$\times \{[\rho J_1(\rho R) + h J_0(\rho R)]^2 + [\rho Y_1(\rho R) + h Y_0(\rho R)]^2\}^{-\frac{1}{2}}.$$

5. A finite body ($R_1 \leq \xi \leq R_2$):

a) any combination of boundary conditions of first, second and third kinds (with the exception of conditions of second kind at both ends of the interval)

$$G(\xi, \xi_0, \tau - t) = \frac{1}{(2\pi)^i} \sum_{n=1}^{\infty} \frac{U_0\left(\mu_n \frac{\xi}{R}\right) U_0\left(\mu_n \frac{\xi_0}{R}\right)}{\| U_0 \|^2} e^{-\frac{\mu_n^2 a}{R^2}(\tau - t)};$$

b) boundary conditions of second kind

$$G(\xi, \xi_0, \tau - t) = \frac{1}{(2\pi)^i} \lim_{\mu_n \to 0} \frac{U_0\left(\mu_n \frac{\xi}{R}\right) U_0\left(\mu_n \frac{\xi_0}{R}\right)}{\| U_0 \|^2} +$$
$$+ \frac{1}{(2\pi)^i} \sum_{n=1}^{\infty} \frac{U_0\left(\mu_n \frac{\xi}{R}\right) U_0\left(\mu_n \frac{\xi_0}{R}\right)}{\| U_0 \|^2} e^{-\frac{\mu_n^2 a}{R^2}(\tau - t)}.$$

In the last two formulas $U_0(\mu_n\frac{\xi}{R})$ is the Eigenfunction of the problem;

$\|U_0\|^2 = \int_{R_1}^{R_2} \xi U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$ is the square of the norm of the Eigenfunction; μ_n is the root of the characteristic equation of the problem; R is the characteristic dimension of the body.

The Eigenfunctions $U_0(\mu_n\frac{\xi}{R})$, characteristic equations and squares of the norm $||U_0||^2$ for a wall $(0 \leq x \leq R)$, solid cylinder $(0 \leq r \leq R)$ and hollow cylinder $(R_1 \leq r \leq R_2)$ with various boundary conditions are presented in Chapter 4.

Example. Establish the temperature field in a system consisting of a "stratified concrete block plus base" on the assumption that the heat-physical characteristics of the concrete and base are identical. The initial temperature is t_0 in the block $(0 \leq x \leq R)$ and zero in the base $(R < x < \infty)$. At the surface of the block, a constant temperature T_1 is fixed. Due to hydration of the cement, heat is liberated in the block, the intensity of heat liberation depending exponentially on time.

The mathematical formulation of the problem

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} + \frac{1}{c\gamma} q_0 \varepsilon(x)\, e^{-m\tau} \quad (0 < x < \infty,\ \tau > 0);$$
$$T(x, 0) = T_0 \varepsilon(x) \quad (0 \leqslant x < \infty);$$
$$T(0, \tau) = T_1,$$

where

$$\varepsilon(x) = \begin{cases} 1, & 0 < x < R; \\ 0, & R < x < \infty. \end{cases}$$

According to formula (3-152)

$$T(x, \tau) = \int_0^\infty T(x_0, 0)\, [G]_{t=0}\, dx_0 - \int_0^\tau \left[G\frac{\partial T}{\partial x_0} - T\frac{\partial G}{\partial x_0}\right]_{x_0=0} dt +$$
$$+ \frac{1}{c\gamma} q_0 \int_0^\tau \int_0^\infty G e^{-mt} \varepsilon(x_0)\, dt\, dx_0,$$

where the Green function G is equal to:

$$G(x, x_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x-x_0)^2}{4a(\tau-t)}\right] - \right.$$
$$\left. - \exp\left[-\frac{(x+x_0)^2}{4a(\tau-t)}\right]\right\}.$$

Let us calculate the integrals of the right portion of the expression for the temperature function T(x, τ).

1. The integral

$$I_1 = \int_0^\infty T(x_0, 0)\,[G]_{t=0}\,dx_0 = T_0 \int_0^R \frac{1}{2\sqrt{\pi a\tau}}\left[e^{-\frac{(x-x_0)^2}{4a\tau}} - e^{-\frac{(x+x_0)^2}{4a\tau}}\right]dx_0$$

is taken by means of the permutations

$$\frac{(x-x_0)^2}{4a\tau} = \xi^2; \quad \frac{(x+x_0)^2}{4a\tau} = \eta^2.$$

The result

$$I_1 = T_0\left\{-\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}}\right] + \frac{1}{2}\operatorname{erfc}\left[\frac{x-R}{2\sqrt{a\tau}}\right] + \frac{1}{2}\operatorname{erfc}\left[\frac{x+R}{2\sqrt{a\tau}}\right]\right\}.$$

2. Since

$$[G]_{x_0=0} = 0 \quad \text{и} \quad T(0, \tau) = T_1,$$

then

$$\left[G\frac{\partial T}{\partial x_0} - T\frac{\partial G}{\partial x_0}\right]_{x_0=0} = -T_1\left[\frac{\partial G}{\partial x_0}\right]_{x_0=0} =$$
$$= -T_1\frac{x}{2a\sqrt{\pi a}}\,\frac{1}{(\tau-t)^{3/2}}\exp\left[-\frac{x^2}{4a(\tau-t)}\right]$$

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and

$$I_2 = -a \int_0^\tau \left[G \frac{\partial T}{\partial x_0} - T \frac{\partial G}{\partial x_0} \right]_{x_0=0} =$$

$$= T_1 \frac{x}{2\sqrt{\pi a}} \int_0^\tau (\tau - t)^{-3/2} \exp\left[-\frac{x^2}{4a(\tau - t)} \right] dt.$$

We know that (see § 5-2)

$$\int_0^\tau \frac{e^{-\frac{x^2}{4a\theta}}}{\theta^{3/2}} d\theta = \frac{2\sqrt{\pi a}}{x} \operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}} \right].$$

Therefore

$$I_2 = T_1 \operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}} \right].$$

3. Considering the result produced we find

$$I_3 = \int_0^\tau \int_0^\infty G e^{-mt} \varphi(x_0)\, dt\, dx_0 = \int_0^\tau e^{-mt} dt \int_0^\infty \frac{1}{2\sqrt{\pi a(\tau - t)}} \times$$

$$\times \left\{ \exp\left[-\frac{(x - x_0)^2}{4a(\tau - t)} \right] - \exp\left[-\frac{(x + x_0)^2}{4a(\tau - t)} \right] \right\} dx_0 =$$

$$= \int_0^\tau e^{-mt} \left\{ -\operatorname{erfc}\left[\frac{x}{2\sqrt{a(\tau - t)}} \right] + \frac{1}{2} \operatorname{erfc}\left[\frac{x - R}{2\sqrt{a(\tau - t)}} \right] + \right.$$

$$\left. + \frac{1}{2} \operatorname{erfc}\left[\frac{x + R}{2\sqrt{a(\tau - t)}} \right] \right\} dt = e^{-m\tau} \int_0^\tau e^{m\theta} \left\{ -\operatorname{erfc}\left[\frac{x}{2\sqrt{a\theta}} \right] + \right.$$

$$\left. + \frac{1}{2} \operatorname{erfc}\left[\frac{x - R}{2\sqrt{a\theta}} \right] + \frac{1}{2} \operatorname{erfc}\left[\frac{x + R}{2\sqrt{a\theta}} \right] \right\} d\theta.$$

But

$$e^{-m\tau} \int_0^\tau e^{m\theta} \operatorname{erfc}\left[\frac{z}{2\sqrt{a\theta}} \right] d\theta = \frac{1}{m} \operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau}} \right] - \frac{1}{m} I_{3/2}\left(m, -\frac{z^2}{4a}, \tau \right).$$

where

$$I_{3/2}\left(m, -\frac{z^2}{4a}, \tau\right) = \frac{z}{2a}\int_0^{\tau} \theta^{-3/2} e^{m(\theta-\tau) - \frac{z^2}{4a\theta}} d\theta.$$

The method of calculation of the integral $I_{3/2}(b, -c^2, \tau)$ will be noted in § 5-2.

Consequently,

$$I_1 = \frac{1}{m}\left\{-\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}}\right] + \frac{1}{2}\operatorname{erfc}\left[\frac{x-R}{2\sqrt{a\tau}}\right] + \frac{1}{2}\operatorname{erfc}\left[\frac{x+R}{2\sqrt{a\tau}}\right]\right\} +$$
$$+\frac{1}{m}\left\{I_{3/2}\left(m, -\frac{x^2}{4a}, \tau\right) - \frac{1}{2} I_{3/2}\left(m, -\frac{(x-R)^2}{4a}, \tau\right) -\right.$$
$$\left. -\frac{1}{2} I_{3/2}\left(m, -\frac{(x+R)^2}{4a}, \tau\right)\right\}$$

Then the solution of the problem is the sum

$$T(x, \tau) = I_1 + I_2 + \frac{q_0}{c\gamma} I_3,$$

where I_1, I_2, I_3 are defined by the expressions presented above.

3-6. Method of Finite Differences

Basic Concepts

The analytic methods presented in § 3-2 - 3-5 can be used to solve linear problems of heat conductivity for bodies limited by coordinate surfaces.

The method of finite differences is a universal and very effective method for approximate numerical solution of linear and nonlinear problems of heat conductivity for bodies of complex shape limited by arbitrary surfaces. Its popularity has been increased by the introduction of computers to the practice of computation.

The essence of the method can be explained on the example of solution of the following one-dimensional problem:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} + Q(x, \tau) \; (0 < x < R, \; 0 < \tau < \theta, \; a = \text{const} > 0); \quad (3\text{-}165)$$

$$T(x, 0) = f(x) \quad (0 \leqslant x \leqslant R); \quad (3\text{-}166)$$

$$T(0, \tau) = \varphi_1(\tau), \;\; T(R, \tau) = \varphi_2(\tau). \quad (3\text{-}167)$$

Area $0 \leq x \leq R$ of continuous change of argument x is divided by points $x_i =$ $= ih$ ($i = 0, 1, \ldots, n$) into n equal sectors of length h; similarly, the interval $0 \leq \tau \leq \theta$ is divided into m equal parts of length ℓ, so that $\tau_k = k\ell$ ($k = 0, 1, \ldots, m$).

Points with the coordinates (x_i, τ_h) are called nodes, their set $\omega_{h\ell} = \{x_i = ih, \tau_k = k\ell; i = 0, 1, \ldots, n, k = 0, 1, \ldots, m\}$ a grid, sectors h and ℓ -- steps in the grid $\omega_h = \{x_i = ih, i = 0, 1, \ldots, n\}$ and $\omega_\ell = \{\tau_k = k\ell, k = 0, 1, \ldots, m\}$ respectively, while the function $T_{i,k} =$ $= T(x_i, \tau_k)$, yielding the temperature at the nodes of the grid is called the grid function.

The derivatives included in differential equation (3-165) can be approximated by finite difference relationships by various methods.

The simplest approximation of the first derivative, for example with respect to x at point x_i in the spatial grid ω_h will be provided by the expressions:

$$\left(\frac{dT}{dx}\right)_i \sim \frac{T_{i+1} - T_i}{h} \quad \text{(right difference derivative);}$$

$$\left(\frac{dT}{dx}\right)_i \sim \frac{T_i - T_{i-1}}{h} \quad \text{(left difference derivative);}$$

$$\left(\frac{dT}{dx}\right)_i \sim \frac{T_{i+1} - T_{i-1}}{2h} \quad \text{(central or bilateral difference derivative).}$$

In order to approximate the second derivative $(d^2T/dx^2)_i$ of two points (x_i, x_{i+1}), (x_{i-1}, x_i) or (x_{i-1}, x_{i+1}) are insufficient; we require at least three points (x_{i-1}, x_i, x_{i+1}).

Then, for example

$$\left(\frac{d^2T}{dx^2}\right)_i \sim \frac{1}{h}\left[\frac{T_{i+1} - T_i}{h} - \frac{T_i - T_{i-1}}{h}\right] = \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}.$$

In the relationships presented, the approximation was performed in the one-dimensional spatial grid ω_h.

Let us analyze the expression

$$\left(\frac{\partial T}{\partial \tau}-\frac{\partial^2 T}{\partial x^2}\right), \tag{3-168}$$

in which the temperature $T(x, \tau)$ is a function of two arguments x and τ, changing in the area $D = \{0 < x < R,\ 0 < \tau \leq \theta)$. The approximation in this case should be performed in the spatial grid $\overline{\omega}_{h\ell}$.

The difference analogue of expression (3-168) is defined as

$$\left(\frac{\partial T}{\partial \tau}-\frac{\partial^2 T}{\partial x^2}\right)\sim\frac{T_{i,k+1}-T_{i,k}}{l}-\frac{T_{i+1,k}-2T_{i,k}+T_{i-1,k}}{h^2}. \tag{3-169}$$

As we can easily see, the approximation with respect to x here is conducted in the lower time layer (k)[1]. It is also possible to approximate in the layer (k + 1), i.e., an approximation of the form

$$\left(\frac{\partial T}{\partial \tau}-\frac{\partial^2 T}{\partial x^2}\right)\sim\frac{T_{i,k+1}-T_{i,k}}{l}-\frac{T_{i+1,k+1}-2T_{i,k+1}+T_{i-1,k+1}}{h^2}. \tag{3-170}$$

Linear combination of expression (3-169) and (3-170) yields

$$\left(\frac{\partial T}{\partial \tau}-\frac{\partial^2 T}{\partial x^2}\right)\sim\frac{T_{i,k+1}-T_{i,k}}{l}-\left[\sigma\frac{T_{i+1,k+1}-2T_{i,k+1}+T_{i-1,k+1}}{h^2}+\right.$$
$$\left.+(1-\sigma)\frac{T_{i+1,k}-2T_{i,k}+T_{i-1,k}}{h^2}\right]; \tag{3-171}$$

where σ is an arbitrary real parameter.

Let us introduce the symbol

$$\Lambda y^s=\frac{y_{i+1,s}-2y_{i,s}+y_{i-1,s}}{h^2}, \tag{3-172}$$

[1]Time layer k or simply layer k will refer to the set of nodes of grid $\omega_{h\ell}$ lying on the line $\tau = \tau_k$.

where

$$s = k,\ k+1.$$

Regardless of the time layer s, the operator inscription of (3-172) is:

$$\Lambda y = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

Sometimes expression (3-171) is called an approximation with weights, σ -- the weight of layer (k + 1).

Based on what we have said, the difference problem corresponding to the continuous problem (3-165)-(3-167) is written as:

$$\frac{T_{i,k+1} - T_{i,k}}{l} = a\,[\sigma\Lambda T^{k+1} + (1-\sigma)\,\Lambda T^{k}\,] + F$$

$$(1 \leqslant i \leqslant n-1,\ 0 \leqslant k \leqslant m-1); \qquad (3\text{-}173)$$

$$T_{i,0} = f(x_i) \quad (0 \leqslant i \leqslant n); \qquad (3\text{-}174)$$

$$T_{0,k} = \varphi_1(\tau_k),\ T_{n,k} = \varphi_2(\tau_k) \quad (1 \leqslant k \leqslant m). \qquad (3\text{-}175)$$

The set of difference equations (3-173)-(3-175), approximating the differential equation, initial and boundary conditions, is called the difference plan.

Term F in the right portion of expression (3-173) can be fixed by various methods, for example $F = F_{i,k} = Q(x_i, \tau_k)$, $F = F_{i,k+1/2} = Q(x_i, \tau_{k+1/2})$ and, etc. Since σ is an arbitrary real number, expressions (3-173) and (3-175) represent a single-parameter set of difference plans.

Let us assume, for example, $\sigma = 0$. Then equation (3-173) becomes:

$$\frac{T_{i,k+1} - T_{i,k}}{l} = a\Lambda T^{k} + F = a\,\frac{T_{i+1,k} - 2T_{i,k} + T_{i-1,k}}{h^2} + F. \qquad (3\text{-}176)$$

From this the grid function $T_{i,k+1}$ is equal to

$$T_{i,k+1} = (1-2M)\,T_{i,k} + M\,(T_{i+1,k} + T_{i-1,k}) + Fl, \qquad (3\text{-}177)$$

where

$$M = \frac{a l}{h^2},$$

i.e., the temperature in the (k + 1)th layer independently at each node x_i of grid $\omega_{h\ell}$ is expressed through the values of temperature at points x_{i-1}, x_i, x_{i+1} in the kth layer. The values of temperature in layer k = 0 are fixed by the initial condition (3-166), the temperature at the ends of the interval x = 0 (i = 0) and x = R (i = n) is known from the boundary conditions (3-167).

Consequently, going over from one layer to another, using formula (3-177) we explicitly define the temperature at all nodes of the calculation area at any moment in time.

The difference plan where $\sigma = 0$ is called an explicit four-point plan.

If $\sigma = 1$, we have the grid equation

$$\frac{T_{i,k+1} - T_{i,k}}{l} = a\frac{T_{i+1,k+1} - 2T_{i,k+1} + T_{i-1,k+1}}{h^2} + F, \tag{3-178}$$

and the grid function in the (k + 1)th layer is established as a result of solution of the system of algebraic equations

$$MT_{i-1,k+1} - (1 + 2M)T_{i,k+1} + MT_{i+1,k+1} = -T_{i,k} - Fl.$$
$$(1 \leqslant i \leqslant n-1,\ 0 \leqslant k \leqslant m-1). \tag{3-179}$$

The values of $T_{i,k}$ and F in the kth layer (where $\tau = \tau_k$) are known. The count moves from layer k to layer (k + 1), beginning with k = 0, for which the initial condition $T_{i,0} = f(x_i)$. The matrix of system (3-179) is a three-diagonal matrix (only the elements along the main and two neighboring diagonals are other than 0). This significantly facilitates the solution of the system. The difference plan where $\sigma = 1$ is called a purely implicit plan with lead.

Finally, where $0 < \sigma < 1$, we produce:

$$\frac{T_{i,k+1} - T_{i,k}}{l} = \sigma \frac{a}{h^2}(T_{i+1,k+1} - 2T_{i,k+1} + T_{i-1,k+1}) + + (1-\sigma)\frac{a}{h^2}(T_{i+1,k} - 2T_{i,k} + T_{i-1,k}) + F. \tag{3-180}$$

The grid function is defined from an algebraic system of equations with a three-diagonal matrix

$$M\sigma T_{i+1,k+1} - (1+2M\sigma)T_{i,k+1} + M\sigma T_{i-1,k+1} = = -[1-2M(1-\sigma)]T_{i,k} - M(1-\sigma)(T_{i+1,k} + T_{i-1,k}) - Fl. \tag{3-181}$$

Plan (3-180) is an implicit, six-point plan with weight σ of the (k + 1)th layer. The most convenient and economical method of solution of equation system (3-181) where $\sigma \neq 0$ is the run-through method. The run-through method is outlined in § 4-2.

Stability of Difference Plans

Above, in place of the continuous problem (3-165)-(3-167), we produced the difference problem (3-173)-(3-175), the solution of which is reduced to the solution of a system of algebraic equations. If this system is not solvable, the difference plan selected should be considered unsuitable.

The second criterion which must be used as a guide in selecting a plan is its stability.

Numerical solution of the algebraic system of equations practically always involves rounding errors in calculation. If the small errors of rounding as the grid is thickened have a tendency to increase, the system is called unstable.

Errors in the calculation can be looked upon as a perturbation of the initial data or the right portion of the equations. In the case of unstable systems, slight changes in input data (right portion of equation, initial and boundary conditions) may lead in a sufficiently fine grid ω_ℓ to arbitrarily large changes in the solution. Therefore, unstable plans cannot be recommended for practical use.

The difference problem is stated correctly (difference plan correct) if its solution T_i with all sufficiently small $h \leq h_0$: 1) exists for arbitrary input data; 2) is continuously dependent on the input data, this dependence being even relative to h.

If the dependence T_i on the input data is even with respect to h, this means that the property of the continuous dependence is retained as $h \to 0$.

The definition of correctness of statement of the difference problem presented above is similar to the definition of correctness of the continuous problem (see Chapter 1).

The property of the continuous dependence of the solution of the difference problem on input data is called stability of the problem (or plan). In the literature, many methods have been described for seeking out necessary and sufficient conditions for stability of difference plans. In most cases, their use requires knowledge of functional analysis. We will limit ourselves here to presentation in simple form of one of these methods.

Let us study the problem of heat conductivity with a homogeneous differential equation

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}\,(0 < x < R,\ 0 < \tau < t),$$

heterogeneous initial condition

$$T(x, 0) = f(x) \quad (0 \leqslant x \leqslant R) \tag{3-182}$$

and homogeneous boundary conditions

$$T(0, \tau) = T(R, \tau) = 0.$$

The corresponding difference problem is written as follows:

$$\begin{gathered}\frac{T_{i,k+1} - T_{i,k}}{l} = a\left[\sigma\Lambda T^{k+1} + (1-\sigma)\Lambda T^{k}\right]\ (1 \leqslant i \leqslant n-1,\ 0 \leqslant k \leqslant m-1);\\ T_{i,0} = f(x_i)\ (0 \leqslant i \leqslant n);\\ T_{0,k} = T_{n,k} = 0\ (1 \leqslant k \leqslant m),\end{gathered} \tag{3-183}$$

where

$$\Lambda T^s = \frac{T_{i+1,s} - 2T_{i,s} + T_{i-1,s}}{h^2}\ (s = k,\ k+1).$$

The solution of the continuous problem (3-182) is:

$$T=\sum_{\nu=1}^{\infty} A_\nu \sin \nu\pi \frac{x}{R} e^{-\nu^2\pi^2\frac{a\tau}{R^2}},$$

where

$$A_\nu=\frac{2}{R}\int_0^R f(x)\sin \nu\pi \frac{x}{R}\,dx.$$

The solution of the difference problem (3-183) is produced by the method of separation of variables.

We assume:

$$T=X(x)\theta(\tau).$$

Then from the first equation in (3-183) it follows that

$$X\frac{\theta_{k+1}-\theta_k}{l}=a\left[\sigma\theta_{k+1}+(1-\sigma)\theta_k\right]\Lambda X,$$

i.e.,

$$\frac{\theta_{k+1}-\theta_k}{al\left[\sigma\theta_{k+1}+(1-\sigma)\theta_k\right]}=\frac{\Lambda X}{X}=-\mu,$$

where μ is a parameter; $\theta_s = \theta(\tau_s)$ $(s = k, k + 1)$.

From this

$$\theta_{k+1}=p\theta_k,$$

where

$$p=\frac{1-(1-\sigma)\,al\mu}{1+\sigma al\mu}. \qquad (3\text{-}184)$$

In order to define function X(x), we use the difference problem of seeking out Eigenvalues (Shturm-Liouville difference problem)

$$\Lambda X(x)+\mu X(x)=0 \ (0<x=ih<R), \ X(0)=X(R)=0. \qquad (3\text{-}185)$$

This problem in index form is written as

$$\frac{X_{i+1}-2X_i+X_{i-1}}{h^2}+\mu X_i=0,$$

or

$$X_{i+1}+X_{i-1}-2\left(1-\frac{1}{2}h^2\mu\right)X_i=0 \ (i=1,\ 2,\ \ldots,\ n-1).$$

The solution of problem (3-185), i.e., the Eigenfunction of the Shturm-Liouville difference problem, is sought in the form

$$X(x)=\sin\alpha x_i,$$

where α is to be defined.

Then from equation (3-185) considering the obvious relationship

$$X_{i+1}+X_{i-1}=\sin\alpha(x_i+h)+\sin\alpha(x_i-h)=2\sin\alpha x_i\cos\alpha h$$

we produce:

$$2\sin\alpha x_i\cos\alpha h=2\left(1-\frac{1}{2}h^2\mu\right)\sin\alpha x_i.$$

Since we are interested in the nontrivial solution, from the last relationship it follows that:

$$\cos\alpha h=1-\frac{1}{2}h^2\mu,$$

from which parameter μ is equal to:

$$\mu = \frac{2}{h^2}(1 - \cos \alpha h) = \frac{4}{h^2} \sin^2 \frac{\alpha h}{2}.$$

The boundary condition at the left end of the interval (x = 0) selected by function X(x) is satisfied identically; from the second boundary condition (where x = R) we produce:

$$\sin \alpha R = 0; \ \alpha_\nu = \nu \frac{\pi}{R} (\nu = 1, 2, \ldots, n-1).$$

Thus, we have found the Eigenvalues of the problem

$$\mu_\nu = \frac{4}{h^2} \sin^2 \frac{\nu \pi h}{2R} \ (\nu = 1, 2, \ldots, n-1)$$

and the corresponding Eigenfunctions

$$X_\nu(x) = \sin \frac{\nu \pi x_i}{R}.$$

The Eigenvalues μ_ν form an increasing sequence of positive numbers

$$0 < \mu_1 < \mu_2 < \ldots < \mu_{n-1},$$

where

$$\mu_1 = \frac{4}{h^2} \sin^2 \frac{\pi h}{2R}; \ \mu_{n-1} = \frac{4}{h^2} \sin^2 \frac{\pi h (n-1)}{2R} = \frac{4}{h^2} \cos^2 \frac{\pi h}{2R}.$$

We can show that the Eigenfunctions $X_\nu(x)$ are orthogonal, i.e.

$$\sum_{\nu=1}^{n-1} X_\nu X_\varkappa h = 0, \text{ если } \nu \neq \varkappa, \qquad (3\text{-}186)$$

Here, the square of the norm $||X_\nu||^2$ is equal to

$$||X_\nu||^2 = \sum_{\nu=1}^{n-1} X_\nu^2 h = \frac{R}{2}. \qquad (3\text{-}186')$$

Function θ_k is defined from formula (3-184)

$$\theta_k = p_\nu \theta_{k-1} = p_\nu^2 \theta_{k-2} = \ldots = p_\nu^k \theta_0,$$

where

$$p_\nu = \frac{1 - (1-\sigma) a l \mu_\nu}{1 + \sigma a l \mu_\nu};$$

θ_0 is a certain constant.

Consequently, the expression

$$(T_{i,k}) = p_\nu^k X_\nu(x_i) = p_\nu^k \sin \frac{\nu \pi x_i}{R}$$

is a particular solution of the first equation of problem (3-183), satisfying the homogeneous boundary conditions.

Let us construct the general solution to problem (3-183) in the form of a sum of particular solutions

$$T_{i,k} = \sum_{\nu=1}^{n-1} a_\nu p_\nu^k \sin \frac{\nu \pi x_i}{R}. \qquad (3\text{-}187)$$

Assuming $\tau = 0$ (i.e., $k = 0$) and considering the initial condition in problem (3-183), we produce:

$$f(x_i) = \sum_{\nu=1}^{n-1} a_\nu \sin \frac{\nu \pi x_i}{R}. \qquad (3\text{-}188)$$

This sum is the expansion of function $f(x_i)$ into a generalized Fourier series with respect to Eigenfunctions of the Shturm-Liouville difference problem (3-185).

In order to determine the Fourier coefficients a_ν, we multiply the right and left portions of formula (3-188) by $\sin \frac{\kappa \pi x_i}{R}$ and add from $\nu = 1$ to $n = 1$. Due to the orthogonality of the Eigenfunctions [see formula (3-186)], we easily find

$$a_\nu = \frac{1}{\|X_\nu\|^2} \sum_{\nu=1}^{n-1} f(x_i) \sin \frac{\nu\pi x_i}{R}.$$

Stability of the finite-difference plan (3-183) in question requires that for any values of a_ν (ν = 1, 2,..., n - 1), grid function $T_{i,k}$ be limited as $\tau \to \infty$ ($k \to \infty$). It is sufficient for this that

$$|p_\nu| = \left| \frac{1-(1-\sigma)\, al\mu_\nu}{1+\sigma al\mu_\nu} \right| \leqslant 1$$

or

$$-1 \leqslant \frac{1-(1-\sigma)\, al \frac{4}{h^2} \sin^2 \frac{\nu\pi h}{2R}}{1+\sigma al \frac{4}{h^2} \sin^2 \frac{\nu\pi h}{2R}} \leqslant 1. \tag{3-189}$$

It can be shown that condition (3-189) is fulfilled for all

$$\sigma \geqslant \frac{1}{2} - \frac{h^2}{4al}. \tag{3-190}$$

In the specialized literature [42, 77, 108, 110, 112], it is proven that relationship (3-190) is a sufficient condition for stability based on initial data not only of finite difference plan (3-183), but also for initial data and the right portion of the following plan, which is more general:

$$\frac{T_{i,k+1}-T_{i,k}}{l} = a\,[\sigma\Lambda T^{k+1} + (1-\sigma)\,\Lambda T^k] + F$$
$$(1 \leqslant i \leqslant n-1,\ 0 \leqslant k \leqslant m-1);$$
$$T_{i,0} = f(x_i) \quad (0 \leqslant i \leqslant n);$$
$$T_{0,k} = \varphi_1(\tau_k);\ T_{n,k} = \varphi_2(\tau_k) \quad (1 \leqslant k \leqslant m).$$

corresponding to the continuous problem

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2} + Q(x, \tau);$$
$$T(x, 0) = f(x);$$
$$T(0, \tau) = \varphi_1(\tau),\ T(R, \tau) = \varphi_2(\tau).$$

Note 1. The partial solutions included with the summation sign in (3-187) are called harmonics, while p_ν is called the transition factor. This is a coefficient which characterizes the attenuation of amplitude of the νth harmonic (since upon transition from the kth layer to the (k + 1)th layer, the amplitude of the νth harmonic is multiplied by p). Spectral stability (stability at each harmonic) indicates stability of the entire plan as a whole.

Note 2. Plans which are stable with all values of h and ℓ are called absolutely stable; plans which are unstable with any values of h and ℓ are called absolutely unstable. Plans which are stable with any limitations placed on h and ℓ are called conditionally stable.

Let us analyze certain plans in more detail.

1. The explicit four-point plan, $\sigma = 0$. The stability condition

$$M = \frac{al}{h^2} \leqslant \frac{1}{2}.$$

Consequently, time step ℓ should be selected so that

$$l \leqslant \frac{h^2}{2a}.$$

Suppose

$$M = \frac{1}{2}, \text{ i. e., } l = \frac{h^2}{2a}.$$

in this case, the time step for the explicit plan is at its maximum, while grid function (3-177) is at its simplest

$$T_{i,k+1} = \frac{T_{i+1,k} + T_{i-1,k}}{2} + Fl.$$

This plan was suggested by Shmidt and carries his name; it is frequently used in various versions of approximate graphic methods for construction of temperature fields.

If

$$M=\frac{1}{3}, \text{ i.e., } l=\frac{h^2}{3a},$$

then

$$T_{i,k+1}=\frac{T_{i+1,k}+T_{i,k}+T_{i-1,k}}{3}+Fl.$$

Finally, where M = 1/6

$$T_{i,k+1}=\frac{T_{i+1,k}+4T_{i,k}+T_{i-1,k}}{6}+Fl.$$

This plan has higher accuracy (see below).

2. Implicit plans with $\sigma \geq 1/2$ are stable for any h and ℓ.

Where $\sigma = 1$, we produce an absolutely stable implicit plan with lead.

The implicit plan with $\sigma = 1/2$, the difference equation for which is

$$\frac{T_{i,k+1}-T_{i,k}}{l}=a\,\frac{\Delta T^{k+1}+\Delta T^{k}}{2}+F,$$

is absolutely stable and has high accuracy, and is sometimes called the Crank-Nicholson plan.

3. Implicit plans with $0 \leq \sigma \leq 1/2$ with σ independent of $M = a\ell/h^2$ are conditionally stable where

$$M=\frac{al}{h^2}\leqslant\frac{1}{2-4\sigma}, \text{ i.e., } l\leqslant\frac{h^2}{a(2-4\sigma)}.$$

Approximation Error. Accuracy and Convergence

For simplicity of presentation of the material, let us use operator inscription. Suppose L is a differential operator, while L_s is its difference analogue (difference operator). The quantity

$$\psi(\zeta)=L_sT(\zeta)-LT(\zeta)$$

is called the error of the difference approximation at point ζ.

It is obvious that

$$\psi(x) = L_h T(x) - LT(x); \quad \psi(\tau) = L_l T(\tau) - LT(\tau),$$

$\psi(x, \tau) = L_{h\ell} T(x, \tau) - LT(x, \tau)$ are the errors in difference approximation at space point x, time point τ, at space-time point (x, τ) respectively. It is said that L_s approximates differential operator L with order $p > 0$ at point ζ if

$$\psi(\zeta) = O(s^p),$$

where $O(s^p)$ is a quantity for which $\lim\limits_{s \to 0} O(s^p)/s^p = C$ (const).

Let us expand the function $T(\zeta)$ in the vicinity of point ζ using the Taylor formula

$$T(\zeta \pm s) = T(\zeta) \pm sT'(\zeta) + \frac{s^2}{2} T''(\zeta) + O(s^3),$$

Then, as we can easily see

$$\frac{T(\zeta + s) - T(\zeta)}{s} = T'(\zeta) + \frac{s}{2} T''(\zeta) + O(s^2); \tag{3-191}$$

$$\frac{T(\zeta) - T(\zeta - s)}{s} = T'(\zeta) - \frac{s}{2} T''(\zeta) + O(s^2); \tag{3-192}$$

$$\frac{T(\zeta + s) - T(\zeta - s)}{2s} = T'(\zeta) + O(s^2); \tag{3-193}$$

$$\frac{T(\zeta + s) - 2T(\zeta) + T(\zeta - s)}{s^2} = T''(\zeta) + \frac{s^2}{12} T'''(\zeta) + O(s^3). \tag{3-194}$$

We can see from this that the error in approximation of the first derivative by the right and left difference derivatives (3-191) and (3-192) has order O(s), the central derivative (3-193) -- $O(s^2)$; the error in approximation of the second derivative of plan (3-194) is $O(s^2)$; the error in approximation of differential operator (3-168) by plans (3-169) and (3-170) is $O(h^2 + \tau)$.

Up to this point we have spoken of an even grid, i.e., a grid with even steps h and ℓ ($x_i = ih$, $\tau = k\ell$).

A grid in which steps h or ℓ are not equal ($h_i \neq h_{i+1}$ or $\ell_k \neq \ell_{k+1}$ with any one or more values of i or k) is called uneven.

For an uneven grid, the approximation error is defined as

$$\max_{1<i<n} |\psi_i| = \max_{1<i<n} |L_h T_i - L T_i|.$$

Suppose T is a continuous solution, while $T_{i,k}$ is the solution of the difference problem.

It is stated that:

1) $T_{i,k}$ converges with T where $h \to 0$ and $\tau \to 0$, if

$$\max_{0 \leq \tau_k \leq t} \sum_{i=1}^{n-1} h_i |T_{i,k} - T| \to 0 \text{ where } h \to 0 \text{ and } \tau \to 0;$$

2) Plan (3-173)-(3-175) converges at rate $O(h^p + \tau^q)$, $p > 0$, $q > 0$ or has accuracy of order $O(h^p + \tau^q)$ if with sufficiently small $h \leq h_0$ and $\tau \leq \tau_0$

$$\max_{0 \leq \tau_k \leq t} \sum_{i=1}^{n-1} h_i |T_{i,k} - T| \leqslant M (h^p + \tau^q); \; M = \text{const} > 0.$$

A theorem is known, which affirms that stability with respect to the right portion and approximation of plan (3-173)-(3-175) indicates even convergence, the order of accuracy coinciding with the order of approximation.

In other words, for plan (3-173)-(3-175), it is correct to state that

$$\max_{1 \leq i \leq n} |T_{i,k} - T| \leqslant M (h^2 + \tau^{q_\sigma}), \; \sigma \neq \sigma_*,$$

where

$$q_\sigma = 1 \text{ where } \sigma \neq \frac{1}{2} \text{ и } F = F_{i,k};$$

$$q_\sigma = 2 \text{ where } \sigma = \frac{1}{2} \text{ и } F = F_{i,k+1/2}$$

(symmetrical six-point Crank-Nicholson plan).

If

$$\sigma = \sigma_* = \frac{1}{2} - \frac{h^2}{12a\tau}, \text{ a } F = F_{i,k+1/2} + \frac{h^2}{12}\left(\frac{\partial^2\theta}{\partial x^2}\right)_{i,k+1/2},$$

then the accuracy of the plan $O(h^4 + \tau^4)$.

Problem of Heat Conductivity with Boundary Conditions of the Third Kind

Up to now, we have solved the problem of heat conductivity with boundary conditions of the first kind. Here, in the selected finite-difference grid, the boundary conditions are precisely satisfied.

The situation is somewhat more complex in the case of boundary conditions of the third kind.

The problem of heat conductivity with boundary conditions of the third kind can be formulated as:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} + Q(x, \tau) \quad (0 < x < R,\ 0 < \tau \leq \theta); \tag{3-195}$$

$$T(x, 0) = f(x) \quad (0 \leq x \leq R); \tag{3-196}$$

$$\frac{\partial T(0, \tau)}{\partial x} = -\beta_1[\psi_1(\tau) - T(0, \tau)];$$

$$\frac{\partial T(R, \tau)}{\partial x} = \beta_2[\psi_2(\tau) - T(R, \tau)]. \tag{3-197}$$

The approximation of differential equation (3-195) yielding a grid function for internal points in the area (0, R) was studied in the previous sections.

The grid function for points at the boundary of interval x = 0 and x = R can be produced by various methods.

1. The derivatives in the left portions of boundary conditions (3-197) can be approximated by simple expressions:

$$\left(\frac{\partial T(0, \tau)}{\partial x}\right)_{0,k+1} \sim \frac{T_{1,k+1} - T_{0,k+1}}{h};$$

$$\left(\frac{\partial T(R, \tau)}{\partial x}\right)_{n,k+1} \sim \frac{T_{n,k+1} - T_{n-1,k+1}}{h}.$$

From this we find:

$$T_{0,k+1}=\frac{1}{1+\beta_1 h}T_{1,k+1}+\frac{\beta_1 h}{1+\beta_1 h}\psi_1(\tau_{k+1});$$

$$T_{n,k+1}=\frac{1}{1+\beta_2 h}T_{n-1,k+1}+\frac{\beta_2 h}{1+\beta_2 h}\psi_2(\tau_{k+1}). \quad (3\text{-}198)$$

The grid functions (3-198) produced have order O(h), whereas the order of approximation of differential equation (3-195) is $O(h^2 + \tau)$. One possible means of increasing the accuracy of approximation of boundary conditions is to draw in additional, above the minimum two, nodes in the grid.

2. In constructing grid functions of increased accuracy, we will base ourselves on certain statements of the theory of interpolation [53].

The Newton formula for forward interpolation of function f(x) assigned for equally separated values of the argument

$$a_0=a;\ a_1=a+h;\ a_2=a+2h,\ \ldots,$$

is:

$$f(x)=f(a)+\frac{x-a}{1!h}\Delta f(a)+\frac{(x-a)(x-a-h)}{2!h^2}\Delta^2 f(a)+$$
$$+\frac{(x-a)(x-a-h)(x-a-2h)}{3!h^3}\Delta^3 f(a)+\ldots \quad (3\text{-}199)$$

Here

$$\Delta f(a)=f(a+h)-f(a)$$

is the first order difference,

$$\Delta^2 f(a)=\Delta f(a+h)-\Delta f(a)=f(a+2h)-2f(a+h)+f(a)$$

is the second order difference,

$$\Delta^3 f(a)=\Delta^2 f(a+h)-\Delta^2 f(a)=f(a+3h)-3f(a+2h)+3f(a+h)-f(a)$$

is the third order difference, etc.

Differentiating expression (3-199) with respect to x and assuming x = a, we find:

$$f'(a)=\frac{1}{h}\left[\Delta f(a)-\frac{1}{2}\Delta^2 f(a)+\frac{1}{3}\Delta^3 f(a)-\ldots\right].$$

Similarly, we can produce the following formula for backward interpolation:

$$f'(a)=\frac{1}{h}\left[\Delta f(a-h)+\frac{1}{2}\Delta^2 f(a-h)+\frac{1}{3}\Delta^3 f(a-3h)+\ldots\right].$$

Limiting ourselves to the first and second differences in the corresponding interpolation formulas, we can write:

$$\left(\frac{\partial T(0,\tau)}{\partial x}\right)_{0,k+1}\sim\frac{1}{2h}\left[-3T_{0,k+1}+4T_{1,k+1}-T_{2,k+1}\right];$$

$$\left(\frac{\partial T(R,\tau)}{\partial x}\right)_{n,k+1}\sim\frac{1}{2h}\left[3T_{n,k+1}-4T_{1,k+1}+T_{n-2,k+1}\right]. \qquad (3\text{-}200)$$

Then the grid functions at the boundary of the interval take on the form:

$$T_{0,k+1}=\frac{2}{3+2\beta_1 h}\psi_1(\tau_{k+1/2})+\frac{4}{3+2\beta_1 h}T_{1,k+1}-\frac{1}{3+2\beta_1 h}T_{2,k+1};$$

$$T_{n,k+1}=\frac{2}{3+2\beta_2 h}\psi_2(\tau_{k+1/2})+\frac{4}{3+2\beta_2 h}T_{n-1,k+1}-\frac{1}{3+2\beta_2 h}T_{n-2,k+1}. \qquad (3\text{-}201)$$

Grid functions (3-201) have order $O(h^2)$.

3. In order to approximate the boundary conditions of the third kind, we frequently use finite-difference approximations produced as a result of application of the method of thermal balances to the elements of the body surface.

The thermal balance equation for the shaded element of a volume near the boundary, for example x = 0 (Figure 3-3), in the case of convective

heat exchange of the body with the environment (boundary conditions of third kind) is:

$$c\gamma\frac{h}{2}\frac{T_{0,k+1}-T_{0,k}}{l}-c\gamma\frac{h}{2}F_{0,k+1/2}=$$
$$=\alpha_1(\psi-T_{0,k})+\lambda\frac{T_{1,k}-T_{0,k}}{h}.$$

Here α_1 is the heat transfer coefficient; λ is the heat conductivity coefficient (we note that $\beta_1 = \alpha_1/\lambda$).

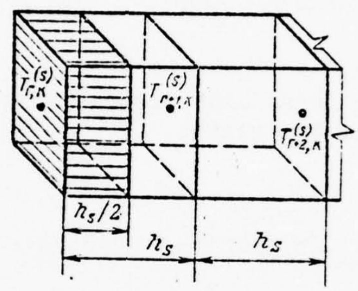

Figure 3-3. Elementary Sector Near Boundary of Body (One-Dimensional Problem)

It follows from this that

$$T_{0,k+1}=(1-2M-2\beta_1 hM)T_{0,k}+$$
$$+2MT_{1,k}+2\beta_1 hM\psi(\tau_{k+1/2})+$$
$$+F_{0,\,k+1/2}l, \tag{3-202}$$

where

$$M=\frac{al}{h^2}.$$

Grid function (3-202) can also be produced on the basis of approximation according to an explicit four-point plan, if outside interval (0, R) at distance h from its boundary x = 0 we place a fictitious node (-1) and determine the temperature in it $T_{-1,k}$ from the boundary condition

$$\frac{T_{1,k}-T_{-1,k}}{2h} = -\beta_1(\psi_1 - T_{0,k}).$$

We also arrive at relationship (3-202) from the following considerations. Based on (3-191) we can write:

$$\frac{T_{1,k}-T_{0,k}}{h} = \frac{\partial T(0,\tau)}{\partial x} + \frac{h}{2}\frac{\partial^2 T(0,\tau)}{\partial x^2} + O(h^2).$$

From the heat conductivity equation (3-195) where x = 0 it follows that

$$\frac{\partial^2 T(0,\tau)}{\partial x^2} = \frac{1}{a}\frac{\partial T(0,\tau)}{\partial \tau} - \frac{1}{a}Q(0,\tau).$$

From this

$$\frac{T_{1,k}-T_{0,k}}{h} - \frac{h}{2}\left[\frac{1}{a}\frac{\partial T(0,\tau)}{\partial \tau} - \frac{1}{a}Q(0,\tau)\right] = \frac{\partial T(0,\tau)}{\partial x} + O(h^2).$$

Thus, the expression in the left portion of the last equation approximates the derivative $\partial T(0, \tau)/\partial x$ with accuracy $O(h^2)$. The time derivative $\partial T(0, \tau)/\partial \tau$ can be replaced with the finite difference relationship

$$\left(\frac{\partial T(0,\tau)}{\partial \tau}\right) \sim \frac{T_{0,k+1}-T_{0,k}}{l}.$$

Then the difference boundary condition where x = 0 is written as

$$\frac{T_{1,k}-T_{0,k}}{h} = \frac{h}{2al}(T_{0,k+1}-T_{0,k}) + \beta_1 T_{0,k} + \frac{h}{2a}F_{0,k+1/2} - \beta_1\psi_{1,k+1/2}, \qquad (3\text{-}203)$$

where

$$F_{0,k+1/2} = Q(0, \tau_{k+1/2}), \quad \psi_{1,k+1/2} = \psi_1(\tau_{k+1/2})$$

and the grid function at boundary x = 0 is reduced to expression (3-202).

Relationship (3-203) approximates the boundary condition of the third kind for the explicit plan.

In the case of an implicit plan with a weight of σ, layer k + 1 of the difference boundary condition of the third kind where x = 0 becomes

$$\sigma\left[\frac{T_{1,k+1}-T_{0,k+1}}{h}-\beta_1 T_{0,k+1}\right]+(1-\sigma)\left[\frac{T_{1,k}-T_{0,k}}{h}\quad\beta_1 T_{0,k}\right]=$$
$$=\frac{h}{2al}(T_{0,k+1}-T_{0,k})-\beta_1\psi_{1,\,k+1/2}-\frac{h}{2a}F_{0,\,k+1/2}. \qquad (3\text{-}204)$$

The difference boundary conditions (3-203) and (3-204) approximate the first boundary condition (3-197) with error $O(h^2)$. The approximation of the second boundary condition (3-197) is performed similarly. As a result where x = R we have:

$$-\sigma\left[\frac{T_{n,k+1}-T_{n-1,\,k+1}}{h}+\beta_2 T_{n,\,k+1}\right]-(1-\sigma)\left[\frac{T_{n,k}-T_{n-1,k}}{h}+\right.$$
$$\left.+\beta_2 T_{n,k}\right]=\frac{h}{2al}(T_{n,k+1}-T_{n,k})+\beta_2\psi_{2,\,k+1/2}-\frac{h}{2a}F_{n,\,k+1/2}.$$

Two-Dimensional and Three-Dimensional Problems

As an example, let us study a two-dimensional problem

$$\frac{\partial T}{\partial\tau}=a\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]+Q(x,\,y,\,\tau)$$
$$(0<x<R,\ 0<y<L,\ 0<\tau\leqslant\theta); \qquad (3\text{-}205)$$
$$T(x,\,y,\,0)=f(x,\,y)\ (0\leqslant x\leqslant R,\ 0\leqslant y\leqslant L); \qquad (3\text{-}206)$$
$$T(0,\,y,\,\tau)=\varphi_1(y,\,\tau)\quad T(R,\,y,\,\tau)=\varphi_2(y,\,\tau); \qquad (3\text{-}207)$$
$$T(x,\,0,\,\tau)=\varphi_3(x,\,\tau)\quad T(x,\,L,\,\tau)=\varphi_4(x,\,\tau).$$

In the area $0 \le x \le R$, $0 \le y \le L$ of continuous change of arguments x and y, two sets of straight lines

$$x=ih_x,\ y=jh_y;\ i=0,\,1,\,2,\,\ldots,\,n,\ j=0,\,1,\,2,\,\ldots,\,s$$

are used to produce the rectangular grid $\omega_{h_x h_y} = \{x_i = ih_x, y_j = jh_y; i = 0, 1, \ldots, n, j = 0, 1, \ldots, s\}$ with steps of h_x and h_y on the x and y coordinates respectively. Let us introduce the space-time grid $\omega_{h_x h_y \ell} = \{x_i = ih_x, y_j = jh_y, \tau_k = k\ell; i = 0, 1, \ldots, n, j = 0, 1, \ldots, s, k = 0, 1, \ldots, m\}$ and the grid function $T_{i,j,k} = T(x_i, y_j, \tau_k)$.

The derivatives included in differential equation (3-205) will be approximated by the finite difference relationships

$$\left(\frac{\partial T}{\partial \tau}\right) \sim \frac{T_{i,j,k+1} - T_{i,j,k}}{l};$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right) \sim \frac{T_{i+1,j,k} - 2T_{i,j,k} + T_{i-1,j,k}}{h_x^2};$$

$$\left(\frac{\partial^2 T}{\partial y^2}\right) \sim \frac{T_{i,j+1,k} - 2T_{i,j,k} + T_{i,j-1,k}}{h_y^2}.$$

This allows the difference problem corresponding to continuous problem (3-205)-(3-207) to be written as:

$$\frac{T_{i,j,k+1} - T_{i,j,k}}{l} = a\left[\frac{T_{i+1,j,k} - 2T_{i,j,k} + T_{i-1,j,k}}{h_x^2} + \right.$$
$$\left. + \frac{T_{i,j+1,k} - 2T_{i,j,k} + T_{i,j-1,k}}{h_y^2}\right] + F,$$
$$(1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant s-1,\ 0 \leqslant k \leqslant m-1);$$
$$T_{i,j,0} = f(x_i, y_j) \quad (0 \leqslant i \leqslant n;\ 0 \leqslant j \leqslant s);$$
$$T_{0,j,k} = \varphi_1(y_j, \tau_k);\ T_{n,j,k} = \varphi_2(y_j, \tau_k);\ T_{i,0,k} = \varphi_3(x_i, \tau_k),$$
$$T_{i,s,k} = \varphi_4(x_i, \tau_k).$$

From this the grid function $T_{i,j,k+1}$ is equal to:

$$T_{i,j,k+1} = (1 - 2M_x - 2M_y)T_{i,j,k} + M_x(T_{i+1,j,k} + T_{i-1,j,k}) +$$
$$+ M_y(T_{i,j+1,k} + T_{i,j-1,k}) + Fl,$$

where

$$M_x = \frac{al}{h_x^2}; \quad M_y = \frac{al}{h_y^2}.$$

Thus, the temperature in the (k + 1)th time layer is expressed independently in each node (x_i, y_j) of grid $\omega_{h_x h_y \ell}$ through the values of the temperature at five points $T_{i-1,j,k}$, $T_{i,j,k}$, $T_{i+1,j,k}$, $T_{i,j-1,k}$, $T_{i,j+1,k}$ in the kth layer. Therefore, the plan in question is called an explicit six-point plan.

If the approximation of second derivatives is conducted in time layer k + 1, we arrive at the difference problem

$$\begin{gathered}
\frac{T_{i,j,k+1} - T_{i,j,k}}{l} = a\Phi T^{k+1} + F \ (1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant s-1, \\
0 \leqslant k \leqslant m-1); \\
T_{i,j,0} = f(x_i, y_j) \quad (0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant s); \\
T_{0,j,k} = \varphi_1(y_j, \tau_k);\ T_{n,j,k} = \varphi_2(y_j, \tau_k); \\
T_{i,0,k} = \varphi_3(x_i, \tau_k);\ T_{i,s,k} = \varphi_4(x_i, \tau_k),
\end{gathered} \tag{3-208}$$

where Φ is an operator defined by the expression

$$\Phi z^r = \frac{z_{i+1,j,r} - 2z_{i,j,r} + z_{i-1,j,r}}{h_x^2} + \frac{z_{i,j+1,r} - 2z_{i,j,r} + z_{i,j-1,r}}{h_y^2}.$$

In this case, the grid function is established as a result of solution of a system of algebraic equations

$$\begin{gathered}
M_x T_{i-1,j,k+1} + M_y T_{i,j-1,k+1} - (1 + 2M_x + 2M_y) T_{i,j,k+1} + \\
+ M_x T_{i+1,j,k+1} + M_y T_{i,j+1,k+1} = -T_{i,j,k} - Fl.
\end{gathered}$$

The difference plan (3-208) is an implicit six-point plan with lead.

If we introduce σ -- the weight of the (k + 1)th layer, we can write the set of difference plans

$$\frac{T_{i,j,k+1}-T_{i,j,k}}{l}=a\left[\sigma\Phi T^{k+1}+(1-\sigma)\Phi T^{k}\right]+Fl;$$

$$T_{i,j,0}=f(x_i,\ y_j);$$

$$T_{0,j,k}=\varphi_1(y_j,\ \tau_k);\quad T_{n,j,k}=\varphi_2(y_j,\ \tau_k);$$

$$T_{i,0,k}=\varphi_3(x_i,\ \tau_k);\quad T_{i,s,k}=\varphi_4(x_i,\ \tau_k).$$

The grid function in this case $T_{i,j,k+1}$ is the solution of a system of algebraic equations

$$\begin{aligned}&M_x\sigma T_{i-1,j,k+1}+M_y\sigma T_{i,j-1,k+1}-(1+2M_x\sigma+2M_y\sigma)T_{i,j,k+1}+\\&+M_x\sigma T_{i+1,j,k+1}+M_y\sigma T_{i,j+1,k+1}=-[1-2M_x(1-\sigma)-\\&-2M_y(1-\sigma)]T_{i,j,k}-M_x(1-\sigma)(T_{i+1,j,k}+T_{i-1,j,k})-\\&-M_y(1-\sigma)(T_{i,j+1,k}+T_{i,j-1,k})-Fl.\end{aligned}$$

Without demonstrating their conclusion, let us simply present the conditions of stability of the plans here studied.

1.

$$0\leqslant\sigma\leqslant\frac{1}{2}.$$

$$M_x+M_y=al\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)\leqslant\frac{1}{2-4\sigma}.$$

In particular, where $\sigma = 0$ (explicit six-point plan)

$$al\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)\leqslant\frac{1}{2}.$$

Consequently, using an explicit six-point plan, the following limitation is placed on the time step ℓ

$$l\leqslant\frac{1}{2a\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)}.$$

2. $\frac{1}{2} < \sigma \leq 1$. The plan is absolutely stable.

The extension of the results produced to the case of the three-dimensional problem of heat conductivity

$$\frac{\partial T}{\partial \tau} = a\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q(x, y, z, \tau)$$
$$(0 < x < R,\ 0 < y < L,\ 0 < z < D,\ 0 < \tau \leqslant \theta);$$
$$T(x, y, z, 0) = f(x, y, z)\ (0 \leqslant x \leqslant R,\ 0 \leqslant y \leqslant L,\ 0 \leqslant z \leqslant D);$$
$$T(0, y, z, \tau) = \varphi_1(y, z, \tau);\ T(R, y, z, \tau) = \varphi_2(y, z, \tau);$$
$$T(x, 0, z, \tau) = \varphi_3(x, z, \tau);\ T(x, L, z, \tau) = \varphi_4(x, z, \tau);$$
$$T(x, y, 0, \tau) = \varphi_5(x, y, \tau);\ T(x, y, D, \tau) = \varphi_6(x, y, \tau)$$

is not difficult.

Let us introduce the space-time grid $\omega_{h_x h_y h_z \ell} = \{x_i = ih_x,\ y_j = jh_y,\ z_p = ph_z;\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, s;\ p = 0, 1, \ldots, r;\ k = 0, 1, \ldots, m\}$, the grid function $T_{i,j,p,k} = T(x_i, y_j, z_p, \tau_k)$, operator G, defined by the expression

$$G\eta^k = \frac{\eta_{i+1,j,p,k} - 2\eta_{i,j,p,k} + \eta_{i-1,j,p,k}}{h_x^2} + \frac{\eta_{i,j+1,p,k} - 2\eta_{i,j,p,k} + \eta_{i,j-1,p,k}}{h_y^2} + \frac{\eta_{i,j,p+1,k} - 2\eta_{i,j,p,k} + \eta_{i,j,p-1,k}}{h_z^2},$$

and the weight of the (k + 1)th time layer σ; then the set of difference plans approximating the continuous problem can be written as

$$\frac{T_{i,j,p,k+1} - T_{i,j,p,k}}{l} = a\left[\sigma G T^{k+1} + (1-\sigma) G T^k\right] + Fl;$$
$$T_{i,j,p,0} = f(x_i, y_j, z_p);$$
$$T_{0,j,p,k} = \varphi_1(y_j, z_p, \tau_k);\ T_{n,j,p,k} = \varphi_2(y_j, z_p, \tau_k);$$
$$T_{i,0,p,k} = \varphi_3(x_i, z_p, \tau_k);\ T_{i,s,p,k} = \varphi_4(x_i, z_p, \tau_k);$$
$$T_{i,j,0,k} = \varphi_5(x_i, y_j, \tau_k);\ T_{i,j,r,k} = \varphi_6(x_i, y_j, \tau_k)$$

Where $\sigma = 0$ we have an explicit eight-point plan, the grid function $T_{i,j,p,k+1}$ is equal to:

$$T_{i,j,p,k+1} = (1 - 2M_x - 2M_y - 2M_z) T_{i,j,p,k} +$$
$$+ M_x (T_{i+1,j,p,k} + T_{i-1,j,p,k}) + M_y (T_{i,j+1,p,k} + T_{i,j-1,p,k}) +$$
$$+ M_z (T_{i,j,p+1,k} + T_{i,j,p-1,k}) + Fl,$$

where

$$M_x = \frac{al}{h_x^2}; \quad M_y = \frac{al}{h_y^2}; \quad M_z = \frac{al}{h_z^2}.$$

If $\sigma \neq 0$, the plan is implicit, the grid function is defined as a result of solution of the system of algebraic equations

$$M_x \sigma T_{i-1,j,p,k+1} + M_y \sigma T_{i,j-1,p,k+1} + M_z \sigma T_{i,j,p-1,k+1} -$$
$$- (1 + 2M_x \sigma + 2M_y \sigma + 2M_z \sigma) T_{i,j,p,k+1} +$$
$$+ M_x \sigma T_{i+1,j,p,k+1} + M_y \sigma T_{i,j+1,p,k+1} + M_z \sigma T_{i,j,p+1,k+1} =$$
$$= -[1 - 2M_x(1-\sigma) - 2M_y(1-\sigma) - 2M_z(1-\sigma)] T_{i,j,p,k} -$$
$$- M_x(1-\sigma)(T_{i+1,j,p,k} + T_{i-1,j,p,k}) - M_y(1-\sigma) \times$$
$$\times (T_{i,j+1,p,k} + T_{i,j-1,p,k}) - M_z(1-\sigma)(T_{i,j,p+1,k} + T_{i,j,p-1,k}).$$

The conditions of stability:

1.

$$M_x + M_y + M_z = al\left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right) \leq \frac{1}{2 - 4\sigma}.$$

In particular, where $\sigma = 0$ (explicit eight-point plan)

$$al\left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right) \leq \frac{1}{2}.$$

2. $\frac{1}{2} < \sigma \leq 1$. The plan is absolutely stable.

CHAPTER 4. CALCULATION OF TEMPERATURE FIELDS OF THE ELEMENTS OF CONCRETE HYDRAULIC STRUCTURES DURING THE PERIOD OF CONSTRUCTION

4-1. Typical Calculation Plans of Structural Elements

Determination of the temperature field of concrete hydraulic structures represents considerable difficulties. Therefore, frequently elements are segregated in the structure and analyzed as bodies of simple geometric form (semilimited body, wall, cylinder, prism, parallelepiped, etc.). The use of such calculation plans significantly simplifies analysis of the temperature state of the structure.

The calculation plan will be called one-dimensional, two-dimensional or three-dimensional if in the area described by it the temperature field is, respectively, one-dimensional, two-dimensional or three-dimensional.

Three-dimensional calculation plans, as a rule, lead to cumbersome algorithms. Therefore, preference should be given to one-dimensional and two-dimensional calculation plans. In each specific case it is desirable to combine them in such a way that the required accuracy of end results is achieved with a smaller volume of computation.

Typical calculation plans of hydraulic structure elements are presented in Figure 4-1.

We present below a brief description of these plans.

One-Dimensional Calculation Plans

Semilimited body. This calculation plan is recommended for determination of the temperature field of bases (rock, old concrete), already present at the moment of beginning of construction of the concrete mass. When this is done, data can be considered concerning the temperatures of the rock base, time of digging of the trench, duration of preparation of the trench for concreting of the blocks. A semilimited body plan can be used in selecting and developing a basis for measures for thermal preparation of the base (steam, hot water, electric heating, etc.), which is particularly important in the construction of hydraulic structures in regions with severe and definitely continental climate (Siberia, Far East, Far North), in the analysis of the temperature mode near the bottom face of the dam, etc.

Unlimited wall. An unlimited wall (or simply wall) will be used to refer to a body in the form of a plane parallel plate, of which two dimensions --

length and height -- are significantly (but less than 4 times) greater than the third dimension -- thickness.

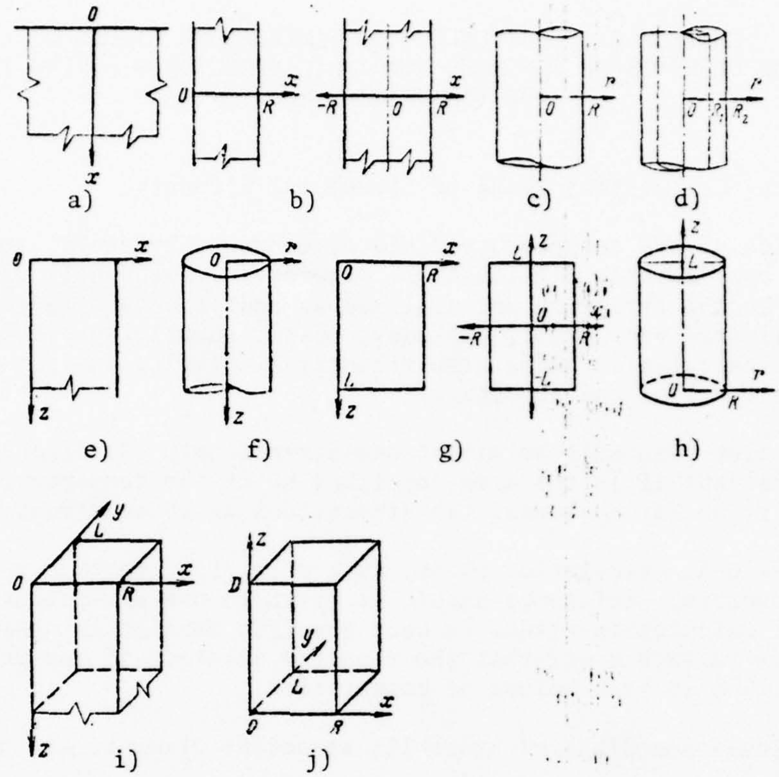

Figure 4-1. Typical Calculation Plans of Elements. a, Semi-limited Body; b, Unlimited Wall; c, Unlimited Solid Cylinder; d, Unlimited Hollow Cylinder; e, Half Strip; f, Semilimited Cylinder; g, Rectangle; h, Finite Cylinder; i, Semilimited Prism of Rectangular Cross Section; j, Parallelepiped

A wall type calculation plan is used in the analysis of temperature fields of concrete wall sections, counterforce dams and flat supporting plates of counterforce dams, individual zones of massive gravity dams (for example caps, etc.), separate walls, walls of sluices, docks and settling pools, etc.

Unlimited cylinder. An unlimited cylinder, solid or hollow (or simply a cylinder) will refer to a body in the form of a circular cylinder, solid or hollow, for which the height is at least 4 times the diameter of the outer cylindrical surface.

A solid cylinder is the calculation plan used for pylons of circular cross section, laboratory concrete specimens; a hollow cylinder is used for the arches of arch and multiple-arch dams, concrete masses with tubular cooling systems, etc.

Two-Dimensional Calculation Plans

Semistrip. A semistrip-type plan can be used to calculate the temperature field of vertical cross sections of individual concrete columns and sections of a dam, the caps of gravity dams, various types of walls in zones adjacent to end surfaces, etc.

A semilimited cylinder is the calculation plan for the end zones of pylons, concrete specimens, etc.

A rectangular calculation plan is used to determine the temperature fields in the horizontal cross section of columns, sections of dams, pylons, etc.

Three-Dimensional Calculation Plans

Characteristic examples of elements of structures in three-dimensional spatial problems are those individual concrete columns which are highest during the process of erection of a dam, the projecting portion of which is comparable to the plan dimensions. We are concerned here with a semilimited column type calculation plan (semilimited rectangular prism).

A three-dimensional temperature field is analyzed using also the parallelepiped type calculation plan.

4-2. Calculations of Temperature Fields of Structural Elements Using One-Dimensional Plans

Semilimited Body ($0 < x < \infty$)

The initial temperature of the body is an arbitrary function of coordinates, the ambient (surface) temperature is a function of time.

The differential equation

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2} \quad (0 < x < \infty, \ \tau > 0); \qquad (4\text{-}1)$$

initial condition

$$T(x, 0) = f(x) \quad (0 \leqslant x < \infty); \qquad (4\text{-}2)$$

boundary conditions of the third kind

$$\frac{\partial T(0, \tau)}{\partial x} = -h[\psi(\tau) - T(0, \tau)], \tag{4-3}$$

of the first kind

$$T(0, \tau) = \varphi(\tau).$$

The solution of such problems using the Green function was described in § 3-5.

It was stated there that

$$T(x, \tau) = \int_0^\infty T(x_0, 0)[G]_{t=0}\,dx_0 - \int_0^\tau \left[G\frac{\partial T}{\partial x_0} - T\frac{\partial G}{\partial x_0}\right]_{x_0=0} dt,$$

where G is the Green function of the corresponding problem, equal to

$$G(x, x_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x-x_0)^2}{4a(\tau - t)}\right] + \exp\left[-\frac{(x+x_0)^2}{4a(\tau - t)}\right]\right\} - h\exp[h^2 a(\tau - t) + h(x + x_0)]\operatorname{erfc}\left[\frac{x + x_0}{2\sqrt{a(\tau - t)}} + h\sqrt{a(\tau - t)}\right]$$

(boundary condition of third kind);

$$G(x, x_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}}\left\{\exp\left[-\frac{(x-x_0)^2}{4a(\tau - t)}\right] - \exp\left[-\frac{(x+x_0)^2}{4a(\tau - t)}\right]\right\}$$ (boundary condition of first kind).

Thus, the temperature field of a semilimited body is described by the expressions:

with boundary conditions of the third kind

$$T(x,\tau)=\int_0^\infty\left\{\frac{1}{2\sqrt{\pi a\tau}}\left(\exp\left[-\frac{(x-x_0)^2}{4a\tau}\right]+\exp\left[-\frac{(x+x_0)^2}{4a\tau}\right]\right)-\right.$$

$$\left.-h\exp[h^2a\tau+h(x+x_0)]\operatorname{erfc}\left[\frac{x+x_0}{2\sqrt{a\tau}}+h\sqrt{a\tau}\right]\right\}f(x_0)\,dx_0+$$

$$+ha\int_0^\tau\left\{\frac{1}{\sqrt{\pi a(\tau-t)}}\exp\left[-\frac{x^2}{4a(\tau-t)}\right]-h\exp[h^2a(\tau-t)+\right.$$

$$\left.+hx]\operatorname{erfc}\left[\frac{x}{2\sqrt{a(\tau-t)}}+h\sqrt{a(\tau-t)}\right]\right\}\psi(t)\,dt; \qquad (4\text{-}4)$$

with boundary conditions of the first kind

$$T(x,\tau)=\frac{1}{2\sqrt{\pi a\tau}}\int_0^\infty\left\{\exp\left[-\frac{(x-x_0)^2}{4a\tau}\right]\right.$$

$$\left.-\exp\left[-\frac{(x+x_0)^2}{4a\tau}\right]\right\}f(x_0)\,dx_0+\frac{x}{2\sqrt{\pi a}}\int_0^\tau\frac{\varphi(t)}{(\tau-t)^{3/2}}\times$$

$$\times\exp\left[-\frac{x^2}{4a(\tau-t)}\right]dt. \qquad (4\text{-}5)$$

With a constant initial temperature $f(x) = T_0$ and constant ambient temperature $\psi(\tau) = T_c$ or surface temperature $\phi(\tau) = T_\pi$

$$\frac{T(x,\tau)-T_0}{T_c-T_0}=\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}}\right]-\exp[h^2a\tau+hx]\times$$

$$\times\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}}+h\sqrt{a\tau}\right]$$ (boundary condition of third kind);

$$\frac{T(x,\tau)-T_0}{T_\pi-T_0}=\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}}\right]$$ (boundary condition of first kind).

Graphs convenient for practical use for determination of the temperature field of a semilimited body (as well as a wall and cylinder) with various initial and boundary conditions are presented in books written by A. I. Pekhovich and V. M. Zhidkikh [87, 88].

Wall $(0 < x < R)$, Solid Cylinder $(0 < r < R)$, Hollow Cylinder $(R_1 < r < R_2)$

In accordance with the results of § 3-3, the analytic solutions of the problem of heat conductivity for these calculation plans can be produced by the method of finite integral transforms of G. A. Greenberg. In the following, these solutions are presented with various forms of the intensity function of heat liberation $q(\tau, T)$ and various initial and boundary conditions. As a rule, they consist of a single formula, suitable for determination of the temperature fields with boundary conditions of the first, second and third kind.

1. The initial temperature is a function of the coordinate. Heat liberation occurs within the body, the intensity of which is described by a generalized function. Boundary conditions of the first, second or third kind, of the same or different types on the different surfaces.

The differential equation

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right)\right] + \frac{1}{c\gamma} q(\tau, T)$$

$$(R_1 < \xi < R_2,\ \tau > 0,\ i = 0 \vee 1); \tag{4-6}$$

initial condition

$$T(\xi, 0) = f(\xi) \quad (R_1 \leq \xi \leq R_2); \tag{4-7}$$

boundary conditions

$$\alpha_j \frac{\partial T}{\partial \xi}\bigg|_\Gamma + \beta_j T|_\Gamma = \gamma_j g_j(\tau) \quad (j = 1,\ 2). \tag{4-8}$$

Here

$$q(\tau, T) = q_\nu (d_\nu + b_\nu T) e^{-m_\nu \tau} \quad (\nu = 1,\ 2, \ldots,\ l),$$

parameter q_ν, d_ν, b_ν, m_ν are fixed in the time sectors $(\tau_{\nu-1}, \tau_\nu)$ where $\tau_0 = 0$.

We recall: for a wall $\xi = x$, $i = 0$, $R_1 = 0$, $R_2 = R$; for a solid cylinder $\xi = r$, $i = 1$, $R_1 = 0$, $R_2 = R$; for a hollow cylinder $\xi = r$, $i = 1$, $R_1 \neq 0$.

Formula (4-8) means that on the surfaces $\xi = R_1$ and $\xi = R_2$, heterogeneous boundary conditions of first, second or third kind are assigned, where the surface temperature, heat flux or ambient temperature are assigned arbitrary functions of time.

The temperature function is equal to[1]:

$$T(\xi, \tau) = \Phi(\xi, \tau) - \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \times$$
$$\times \exp\left[-\mu_n^2 \frac{a\tau_s}{R^2}\right] \exp\left[\sum_{\nu=1}^{s} c_\nu\right] \left\{B_n - \sum_{\nu=1}^{s} \Psi_{\nu n} \exp\left[-\sum_{\nu'=1}^{\nu} c_{\nu'}\right]\right\}. \quad (4\text{-}9)$$

where

$$B_n = \overline{\Phi}_n(0) - \bar{f}_n;$$
$$\Psi_{\nu n} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu}\right] \int_{\tau_{\nu-1}}^{\tau_\nu} \mathscr{L}_\nu(\zeta) \exp\left[\mu_n^2 \frac{a\zeta}{R^2} - m_\nu \zeta + \right.$$
$$\left. + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta}\right] d\zeta;$$
$$\mathscr{L}_\nu(\tau) = \frac{q_\nu d_\nu}{c\gamma} N_n + \frac{q_\nu b_\nu}{c\gamma} \overline{\Phi}_n(\tau) - \overline{\Phi}'_n e^{m_\nu \tau};$$
$$c_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma} \{\exp[-m_\nu \tau_{\nu-1}] - \exp[-m_\nu \tau_\nu]\};$$
$$\bar{f}_n = \int_{R_1}^{R_2} \xi^i f(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$
$$\overline{\Phi}_n(\tau) = \int_{R_1}^{R_2} \xi^i \Phi(\xi, \tau) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$

[1]The boundary conditions of the second kind are not analyzed simultaneously on both surfaces $\xi = R_1$ and $\xi = R_2$.

μ_n is the root of the characteristic equation; $U_0(\mu_n \frac{\xi}{R})$ is the Eigenfunction of the problem; $||U_0||^2$ is the square of the norm of the Eigenfunction;

$\Phi(\xi, \tau)$ is a substitution function; τ_s is the calculation time, related to the sth time interval of the generalized heat liberation intensity function.

The values of $U_0(\mu_n(\xi/R))$, $||U_0||^2$, N_n, μ_n, $\Phi(\xi, \tau)$ depend on the geometric form of the body and the type of boundary conditions and are determined by the reference data presented below.

Thus, for a symmetrically cooled wall ($-R < x < R$) and a circular cross section pylon ($0 < r < R$) with boundary conditions of the third kind[1], constant ambient temperature T_c and constant initial temperature T_0:

$$T(\xi, \tau) = \Phi(\xi) - \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{\xi}{R}\right) \exp\left[-\mu_n^2 \frac{a\tau_s}{R^2}\right] \times$$
$$\times \exp\left[\sum_{\nu=1}^{s} c_\nu\right] \left\{B_n - \sum_{\nu=1}^{s} L_\nu \Psi_{\nu n} \exp\left[-\sum_{\nu'=1}^{\nu} c_{\nu'}\right]\right\}. \quad (4\text{-}10)$$

where

$$\Phi(\xi) = T_c;\ B_n = T_c - T_0;\ L_\nu = \frac{1}{c\gamma}(q_\nu d_\nu + q_\nu b_\nu T_c);$$

$$\Psi_{\nu n} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu}\right] \int_{\tau_{\nu-1}}^{\tau_\nu} \exp\left[\mu_n^2 \frac{a\zeta}{R^2} - m_\nu \zeta + \right.$$
$$\left. + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta}\right] d\zeta,$$

where for the wall μ_n is the root of the characteristic equation

$$\cot \mu_n = \frac{\mu_n}{\mathrm{Bi}};\ \mathrm{Bi} = hR;$$
$$U_0\left(\mu_n \frac{\xi}{R}\right) = \cos \mu_n \frac{x}{R};$$
$$A_n = \frac{N_n}{||U_0||^2} = (-1)^{n+1} \frac{2\,\mathrm{Bi}\sqrt{\mu_n^2 + \mathrm{Bi}^2}}{\mu_n(\mu_n^2 + \mathrm{Bi}^2 + \mathrm{Bi})};$$

---

[1]Boundary conditions:
wall $(\partial T(0, \tau))/\partial x = 0$ -- condition of symmetry, $(\partial T(R, \tau))/\partial x = h[T_c - T(R, \tau)]$; pylon $(\partial T(R, \tau))/\partial r = h[T_c - T(R, \tau)]$, $(\partial T(0, \tau))/\partial r = 0$, $T(0, \tau) \neq \infty$ (condition of limited temperature function).

For a circular cross section pylon μ_n is the root of the characteristic equation

$$\frac{J_0(\mu_n)}{J_1(\mu_n)} = \frac{\mu_n}{\mathrm{Bi}}; \quad \mathrm{Bi} = hR;$$

$$U_0\left(\mu_n \frac{\xi}{R}\right) = J_0\left(\mu_n \frac{r}{R}\right);$$

$$A_n = \frac{2\mathrm{Bi}}{J_0(\mu_n)(\mu_n^2 + \mathrm{Bi}^2)}.$$

For an asymmetrically cooled wall ($0 < x < R$) with boundary conditions of the third kind

$$\frac{\partial T(0, \tau)}{\partial x} = -h_1[T_1 - T(0, \tau)]; \quad \frac{\partial T(R, \tau)}{\partial x} = h_2[T_2 - T(R, \tau)]$$

the temperature function has the same form as in (4-10). However

$$\Phi(\xi) = T_2 + \frac{(T_2 - T_1)\mathrm{Bi}_1}{(\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1\mathrm{Bi}_2)}\left(1 + \mathrm{Bi}_2 - \mathrm{Bi}_2 \frac{x}{R}\right);$$

$$B_n = (T_2 - T_0)\left[(-1)^{n+1}\mathrm{Bi}_2\sqrt{\frac{\mu_n^2 + \mathrm{Bi}_1^2}{\mu_n^2 + \mathrm{Bi}_2^2}} + \mathrm{Bi}_1\right] -$$

$$- \mathrm{Bi}_1(T_1 - T_2);$$

$$L_v = \frac{q_v}{c\gamma}(d_v + b_v T_2)\left[(-1)^{n+1}\mathrm{Bi}_2\sqrt{\frac{\mu_n^2 + \mathrm{Bi}_1^2}{\mu_n^2 + \mathrm{Bi}_2^2}} + \mathrm{Bi}_1\right] +$$

$$+ \mathrm{Bi}_1 \frac{q_v b_v}{c\gamma}(T_1 - T_2);$$

μ_n is the root of the characteristic equation

$$\cot \mu_n = \frac{\mu_n^2 - \mathrm{Bi}_1\mathrm{Bi}_2}{\mu_n(\mathrm{Bi}_1 + \mathrm{Bi}_2)}; \quad \mathrm{Bi}_1 = h_1 R, \ \mathrm{Bi}_2 = h_2 R;$$

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R};$$

$$A_n = \frac{2}{\mu_n\left[\frac{\mu_n^2 + \mathrm{Bi}_1^2}{\mu_n^2 + \mathrm{Bi}_2^2}(\mu_n^2 + \mathrm{Bi}_2^2 + \mathrm{Bi}_2) + \mathrm{Bi}_1\right]}.$$

In the solutions presented above, we encounter the integrals

$$I_1 = \exp[-a\tau_s - be^{-m\tau_v}] \int_{\tau_{v-1}}^{\tau_v} \exp[a\zeta - m\zeta + be^{-m\zeta}]\, d\zeta;$$

$$I_2 = \exp[-a\tau_s - be^{-m\tau_v}] \int_{\tau_{v-1}}^{\tau_v} \psi(\zeta) \exp[a\zeta - m\zeta + be^{-m\zeta}]\, d\zeta,$$

the calculation of which is usually performed using methods of numerical integration.

Here $\psi(\tau)$ is the ambient (surface) temperature or its derivative; a, b, m are certain constants.

We can, however, suggest another method of calculation, based on representation of these integrals in the form of series.

We know that

$$e^{be^{-m\tau}} = \sum_{p=0}^{\infty} \frac{b^p}{p!} e^{-pm\tau}.$$

Then

$$I_1 = \exp[-be^{-m\tau_v}] \sum_{p=0}^{\infty} \frac{b^p}{p!\,[a-(p+1)m]} \{\exp[-a(\tau_s - \tau_v) - (p+1)m\tau_v] - \exp[-a(\tau_s - \tau_{v-1}) - (p+1)m\tau_{v-1}]\}.$$

We assume

$$\psi(\tau) = \sin(\omega\tau + \varepsilon),$$

which corresponds to description of the temperature of the medium by a harmonic function.

In this case

$$I_2 = \exp\left[-be^{-m\tau_\nu}\right] \sum_{p=0}^{\infty} \frac{b^p}{p!\,\{\omega^2 + [a-(p+1)\,m]^2\}} \{\exp[-a(\tau_s - \tau_\nu) - (p+1)\,m\tau_\nu]\sin(\omega\tau_\nu + \varepsilon - \delta) - \exp[-a(\tau_s - \tau_{\nu-1}) - (p+1)\,m\tau_{\nu-1}]\sin(\omega\tau_{\nu-1} + \varepsilon - \delta)\},$$

where

$$\delta = \operatorname{arctg} \frac{\omega}{a-(p+1)\,m}.$$

2. The initial and boundary conditions are homogeneous, 0. The intensity of heat liberation in the concrete is an exponential function of time.

The differential equation

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right)\right] + \frac{1}{c\gamma}\, q(\tau);$$

initial condition

$$T(\xi,\ 0) = 0;$$

boundary conditions

$$\alpha_j \frac{\partial T}{\partial \xi}\Big|_\Gamma + \beta_j T|_\Gamma = 0 \quad (j = 1,\ 2).$$

Here

$$q = q_0 e^{-m\tau}.$$

Note. Below, in those cases when problems differ only in the form of the heat liberation intensity function q, the differential equation and boundary conditions will not be presented.

The solution can be produced either by means of formula (4-9), or by applying a finite integral transform directly to the problem.

The result

$$T(\xi, \tau) = \frac{q_0 R^2}{\lambda} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 - m^{*2}} U_0\left(\mu_n \frac{\xi}{R}\right) \left[e^{-m\tau} - e^{-\mu_n^2 \frac{a\tau}{R^2}}\right] \tag{4-11'}$$

or

$$T(\xi, \tau) = \frac{q_0 R^2}{\lambda} \left[w(\xi) e^{-m\tau} - \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 - m^{*2}} U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}\right], \tag{4-11}$$

where

$$A_n = \frac{N_n}{\|U_0\|^2}; \; m^{*2} = \frac{mR^2}{a};$$

$w(\xi)$ is a function of the coordinates, the solution of the differential equation

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dw}{d\xi}\right) + \frac{m}{a} w = -1$$

with homogeneous boundary conditions corresponding to the boundary conditions of the initial problem.

Formula (4-11) is produced as a result of summation of the first series of expression (4-11') according to the recommendations of § 3-3.

As an illustration, we present certain values of function $w(\xi)$ for the wall and the cylinder.

a) Wall ($0 \leq x \leq R$). The boundary conditions of the first kind where $x = 0$ and of the third kind where $x = R$

$$\omega(\xi)=\frac{1}{m^{*2}}\left\{[m^*\cos m^*+\mathrm{Bi}_2\sin m^*]^{-1}\left[m^*\cos m^*\left(1-\frac{x}{R}\right)+\right.\right.$$
$$\left.\left.+\mathrm{Bi}_2\sin m^*\left(1-\frac{x}{R}\right)+\mathrm{Bi}_2\sin m^*\frac{x}{R}\right]-1\right\},$$

where

$$m^{*2}=\frac{mR^2}{a};\ \mathrm{Bi}_2=h_2R.$$

b) Solid cylinder ($0 \le r \le R$). Boundary conditions of the third kind

$$\omega(\xi)=\frac{1}{m^{*2}}\left[\frac{\mathrm{Bi}\,J_0\left(m^*\frac{r}{R}\right)}{\mathrm{Bi}\,J_0(m^*)-m^*J_1(m^*)}-1\right];\ \mathrm{Bi}=hR.$$

c) Hollow cylinder ($R_1 \le r \le R_2$). Boundary conditions of the first kind where $r = R_1$ and of the second kind where $r = R_2$

$$\omega(\xi)=\frac{1}{m^{*2}}\left\{[J_0(m^*)Y_1(km^*)-Y_0(m^*)J_1(km^*)]^{-1}\times\right.$$
$$\left.\times\left[Y_1(km^*)J_0\left(m^*\frac{r}{R_1}\right)-J_1(km^*)Y_0\left(m^*\frac{r}{R_1}\right)\right]-1\right\},$$

where

$$m^{*2}=\frac{mR_1}{a};\ k=\frac{R_2}{R_1}.$$

3. The initial and boundary conditions are homogeneous. The intensity of heat liberation is an arbitrary function of time $q = q(\tau)$.

As was indicated in § 2-2, the heat liberation intensity function in the concrete $q(\tau)$, which depends solely on time, can be approximated by two methods:

a) $q = q_\nu e^{-m_\nu \tau}$ $(\nu = 1, 2, \ldots, \ell)$,

where q_ν, m_ν are parameters defined in the time sector $(\tau_{\nu-1}, \tau_\nu)$;

b) $q = \sum_{\nu=1}^{s} q_\nu e^{-\nu m\tau}$.

In case "a" we have:

$$T(\xi, \tau) = \frac{R^2}{\lambda} \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau_s}{R^2}} \sum_{\nu=1}^{s} \frac{q_\nu}{\mu_n^2 - m_\nu^{**}} \times$$
$$\times \left\{ \exp\left[\left(\mu_n^2 \frac{a}{R^2} - m_\nu^*\right)\tau_\nu^*\right] - \exp\left[\left(\mu_n^2 \frac{a}{R^2} - m_\nu^*\right)\tau_{\nu-1}\right]\right\};$$
$$m_\nu^{**} = \frac{m_\nu R^2}{a}.$$

In case "b" the temperature function is a sum, based on the number of terms in the expression for the heat liberation intensity function, of solutions such as (4-11').

4. The initial and boundary conditions are homogeneous. The heat liberation intensity function is constant, $q = q_0$ = const.

The temperature function

$$T(\xi, \tau) = \frac{q_0 R^2}{\lambda} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right) \left[1 - e^{-\mu_n^2 \frac{a\tau}{R^2}}\right] \tag{4-12'}$$

or

$$T(\xi, \tau) = \frac{q_0 R^2}{\lambda} \left[v(\xi) - \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}\right], \tag{4-12}$$

where the function of the coordinates $v(\xi)$ is defined as a result of solution of the differential equation

$$\frac{1}{\xi^i} \frac{d}{d\xi}\left(\xi^i \frac{dv}{d\xi}\right) = -1$$

with homogeneous boundary conditions of the same kind as in the initial problem.

This function, for example, is equal to:

For a wall ($0 \leq x \leq R$) with boundary conditions of the first kind ($x = 0$) and the third kind ($x = R$)

$$v(\xi) = \frac{1}{2}\left[-\frac{x^2}{R^2} + \frac{Bi_2 + 2}{Bi_2 + 1}\frac{x}{R}\right]; \; Bi_2 = h_2 R;$$

For a solid cylinder ($0 \leq r \leq R$) with boundary conditions of the third kind

$$v(\xi) = \frac{1}{4}\left[1 - \frac{r^2}{R^2} + \frac{2}{Bi}\right]; \; Bi = hR;$$

For a hollow cylinder $R_1 \leq r \leq R_2$) with boundary conditions of the first kind ($r = R_1$) and third kind ($r = R_2$)

$$v(\xi) = \frac{1}{4}\left[1 - \frac{r^2}{R_1^2} + \frac{k(k^2 + 2k - Bi_2)}{1 + k\,Bi_2 \ln k}\ln\frac{r}{R_1}\right]; \; Bi_2 = h_2 R_1; \; k = \frac{R_2}{R_1}.$$

5. The initial temperature is a function of the coordinates.

The boundary conditions are heterogeneous, similar to (4-8). The heat liberation in the body is nil, $q = 0$.

The temperature function

$$T(\xi, \tau) = \Phi(\xi, \tau) - \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}} \times$$
$$\times\left[B_n + \int_0^{\tau} \overline{\Phi}'_n(\tau)\, e^{\mu_n^2 \frac{a\tau}{R^2}}\, d\tau\right].$$

The symbols are the same as in problem 1 [formula (4-9)].

If $g_j(\tau) = g_j$ = const in the boundary condition (4-8) and substitution function $\Phi(\xi)$ is used with F functions of the first kind or if $g_j(\tau) = g_{j0}e^{\pm m_j \tau}$ (where m_j may be either real or imaginary) and the substitution function

$\Phi(\xi, \tau)$ is used with F functions of the second kind, the temperature function becomes

$$T(\xi, \tau) = \Phi(\xi, \tau) - \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} [\bar{\Phi}_n - \bar{f}_n] U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}.$$

Reference Data

Eigenfunctions, Characteristic Equations and Related Integrals

We present here the Eigenfunctions $U_0(\mu_n\frac{\xi}{R})$, characteristic equations, the roots of which are the numbers μ_n, squares of the norm of the Eigenfunctions $\|U_0\|^2 = \int_{R_1}^{R_2} \xi^i U_0^2 \left(\mu_n \frac{\xi}{R}\right) d\xi$ and integrals $N_n = \int_{R_1}^{R_2} \xi^i U_0 \left(\mu_n \frac{\xi}{R}\right)$ for a wall, solid cylinder and hollow cylinder with all possible combinations of boundary conditions of the first, second and third kinds on their surfaces. Using these data, one can formally construct formulas for the direct and reverse finite integral transforms of G. A. Greenberg:

$$\bar{u}_n = \int_{R_1}^{R_2} \xi^i u(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$

$$u(\xi) = \sum_{n=1}^{\infty} \frac{\bar{u}_n}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right)$$

or with boundary conditions of the second kind at both ends of the interval $[R_1, R_2]$

$$u(\xi) = \frac{\int_{R_1}^{R_2} \xi^i u(\xi)\, d\xi}{\int_{R_1}^{R_2} \xi^i d\xi} + \sum_{n=1}^{\infty}{}' \frac{\bar{u}_n}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right).$$

Here, as in Table 3-1, the boundary conditions for the wall and hollow cylinder are written with Roman numerals: I (first kind), II (second kind), III (third kind), the first indicating the boundary condition where $\xi = R_1$, the second -- where $\xi = R_2$.

Wall $(0 < x < R)$.

1. Boundary conditions I-I.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\sin \mu_n = 0; \quad \mu_n = n\pi.$$

$$\|U_0\|^2 = \frac{R}{2}; \quad N_n = \frac{2R}{(2n-1)\pi}.$$

2. Boundary conditions II-II.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \cos \mu_n \frac{x}{R}.$$

Characteristic equation

$$\sin \mu_n = 0; \quad \mu_n = n\pi.$$

$$\|U_0\|^2 = \frac{R}{2}; \quad N_n = 0.$$

3. Boundary conditions III-III.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\cot \mu_n = \frac{\mu_n^2 - \mathrm{Bi}_1 \mathrm{Bi}_2}{\mu_n (\mathrm{Bi}_1 + \mathrm{Bi}_2)}; \quad \mathrm{Bi}_1 = h_1 R; \quad \mathrm{Bi}_2 = h_2 R.$$

$$\|U_0\|^2 = \frac{R}{2}\left[\frac{\mathrm{Bi}_2(\mu_n^2 + \mathrm{Bi}_1^2)}{\mu_n^2 + \mathrm{Bi}_2^2} + \mu_n^2 + \mathrm{Bi}_1^2 + \mathrm{Bi}_1\right];$$

$$N_n = \frac{R}{\mu_n}\left[\mathrm{Bi}_1 + (-1)^{n+1} \mathrm{Bi}_2 \sqrt{\frac{\mu_n^2 + \mathrm{Bi}_1^2}{\mu_n^2 + \mathrm{Bi}_2^2}}\right];$$

4. Boundary conditions I-II.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\cos \mu_n = 0, \quad \mu_n = (2n-1)\frac{\pi}{2};$$

$$\|U_0\|^2 = \frac{R}{2}; \quad N_n = \frac{2R}{(2n-1)\pi}.$$

5. Boundary conditions II-I.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \cos \mu_n \frac{x}{R}.$$

Characteristic equation

$$\cos \mu_n = 0; \quad \mu_n = (2n-1)\frac{\pi}{2};$$

$$\|U_0\|^2 = \frac{R}{2}; \quad N_n = \frac{(-1)^{n+1} 2R}{(2n-1)\pi}.$$

6. Boundary conditions I-III.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right)=\sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\tan \mu_n=-\frac{\mu_n}{\mathrm{Bi}_2};\ \mathrm{Bi}_2=h_2R.$$

$$\|U_0\|^2=\frac{R}{2}\,\frac{\mathrm{Bi}_2^2+\mathrm{Bi}_2+\mu_n^2}{\mathrm{Bi}_2^2+\mu_n^2};$$

$$N_n=\frac{R}{\mu_n}\left[1+\frac{(-1)^{n+1}\,\mathrm{Bi}_2}{\sqrt{\mathrm{Bi}_2^2+\mu_n^2}}\right].$$

7. Boundary conditions III-I.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right)=\mu_n \cos \mu_n \frac{x}{R}+\mathrm{Bi}_1 \sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\tan \mu_n=-\frac{\mu_n}{\mathrm{Bi}_1};\ \mathrm{Bi}_1=h_1R.$$

$$\|U_0\|^2=\frac{R}{2}(\mu_n^2+\mathrm{Bi}_1^2+\mathrm{Bi}_1);\ N_n=\frac{R}{\mu_n}[\mathrm{Bi}_1+(-1)^{n+1}\sqrt{\mu_n^2+\mathrm{Bi}_1^2}].$$

8. Boundary conditions II-III.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right)=\cos \mu_n \frac{x}{R}.$$

Characteristic equation

$$\cot \mu_n = \frac{\mu_n}{\mathrm{Bi}_2}; \quad \mathrm{Bi}_2 = h_2 R.$$

$$\| U_0 \|^2 = \frac{R}{2} \frac{\mathrm{Bi}_2^2 + \mathrm{Bi}_2 + \mu_n^2}{\mathrm{Bi}_2^2 + \mu_n^2}; \quad N_n = \frac{(-1)^{n+1} R\,\mathrm{Bi}_2}{\mu_n \sqrt{\mathrm{Bi}_2^2 + \mu_n^2}}.$$

9. Boundary condition III-II.

Eigenfunction

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R}.$$

Characteristic equation

$$\cot \mu_n = \frac{\mu_n}{\mathrm{Bi}_1}; \quad \mathrm{Bi}_1 = h_1 R.$$

$$\| U_0 \|^2 = \frac{R}{2} (\mathrm{Bi}_1^2 + \mathrm{Bi}_1 + \mu_n^2); \quad N_n = \frac{R\,\mathrm{Bi}_1}{\mu_n}.$$

Solid cylinder (0 < r < R).

1. Boundary condition of first kind.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R}\right) = J_0\left(\mu_n \frac{r}{R}\right).$$

Characteristic equation

$$J_0(\mu_n) = 0;$$

$$\| U_0 \|^2 = \frac{R^2}{2} J_1^2(\mu_n); \quad N_n = R^2 J_1(\mu_n).$$

2. Boundary condition of second kind

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R}\right) = J_0\left(\mu_n \frac{r}{R}\right).$$

Characteristic equation

$$J_1(\mu_n) = 0;$$
$$\|U_0\|^2 = \frac{R^2}{2} J_0^2(\mu_n); \quad N_n = 0.$$

3. Boundary condition of third kind.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R}\right) = J_0\left(\mu_n \frac{r}{R}\right).$$

Characteristic equation

$$\frac{J_0(\mu_n)}{J_1(\mu_n)} = \frac{\mu_n}{\mathrm{Bi}}; \quad \mathrm{Bi} = hR;.$$
$$\|U_0\|^2 = \frac{R^2(\mathrm{Bi}^2 + \mu_n^2)}{2\mu_n}; \quad N_n = \frac{R^2\,\mathrm{Bi}}{\mu_n} J_1(\mu_n).$$

Hollow cylinder ($R_1 < r < R_2$).

1. Boundary condition I-I.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_0(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n) = 0; \quad k = \frac{R_2}{R_1}.$$

$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \frac{J_0^2(\mu_n) - J_0^2(k\mu_n)}{J_0^2(k\mu_n)}; \quad N_n = \frac{2R_1^2}{\pi^2 \mu_n^2} \frac{J_0(\mu_n) - J_0(k\mu_n)}{J_0(k\mu_n)}.$$

2. Boundary condition II-II.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$Y_1(\mu_n) J_1(k\mu_n) - J_1(\mu_n) Y_1(k\mu_n) = 0;$$

$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \frac{J_1^2(\mu_n) - J_1^2(k\mu_n)}{J_1^2(k\mu_n)}; \quad N_n = 0.$$

3. Boundary condition III-III.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = \left[Y_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} Y_1(\mu_n)\right] J_0\left(\mu_n \frac{r}{R_1}\right) - \\ - \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right] Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$\mathrm{Bi}_2 \{\mathrm{Bi}_1 [Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n)] + \mu_n [Y_1(\mu_n) J_0(k\mu_n) - \\ - J_1(\mu_n) Y_0(k\mu_n)]\} - \mu_n \{\mathrm{Bi}_1 [Y_0(\mu_n) J_1(k\mu_n) - J_0(\mu_n) Y_1(k\mu_n)] + \\ + \mu_n [Y_1(\mu_n) J_1(k\mu_n) - J_1(\mu_n) Y_1(k\mu_n)]\} = 0; \quad \mathrm{Bi}_1 = h_1 R_1, \quad \mathrm{Bi}_2 = h_2 R_1;$$

$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \left\{ \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n) \right]^{-2} \left[J_0(\mu_n) + \right.\right.$$

$$+\frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\Big]^2 \left(1+\frac{\mu_n^2}{\mathrm{Bi}_2^2}\right) - \left(1+\frac{\mu_n^2}{\mathrm{Bi}_1^2}\right)\Bigg\};$$

$$N_n = \frac{2R_1^2}{\pi\mu_n^2}\left\{\left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-1}\left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right] - 1\right\}.$$

4. Boundary condition I-II.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_0(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$Y_0(\mu_n) J_1(k\mu_n) - J_0(\mu_n) Y_1(k\mu_n) = 0;$$

$$\|U_0\|^2 = \frac{2R_1^2}{\pi^2\mu_n^2} \frac{J_0^2(\mu_n) - J_1^2(k\mu_n)}{J_0^2(k\mu_n)}; \quad N_n = -\frac{2R_1^2}{\pi\mu_n^2}.$$

5. Boundary condition II-I.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$Y_1(\mu_n) J_0(k\mu_n) - J_1(\mu_n) Y_0(k\mu_n) = 0;$$

$$\|U_0\|^2 = \frac{2R_1^2}{\pi^2\mu_n^2} \frac{J_1^2(\mu_n) - J_0^2(k\mu_n)}{J_0^2(k\mu_n)}; \quad N_n = \frac{2R_1^2}{\pi\mu_n^2} \frac{J_1(\mu_n)}{J_0(k\mu_n)}.$$

6. Boundary condition I-III.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_0(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$\mathrm{Bi}_2[Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n)] - \mu_n [Y_0(\mu_n) J_1(k\mu_n) -$$
$$- J_0(\mu_n) Y_1(k\mu_n)] = 0; \quad \mathrm{Bi}_2 = h_2 R_1;$$

$$\|U_0\|^2 = \frac{2R_1^2}{\pi^2\mu_n^2}\left\{\left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-2} J_0^2(\mu_n)\left(1 + \frac{\mu_n^2}{\mathrm{Bi}_2^2}\right) - 1\right\};$$

$$N_n = \frac{2R_1^2}{\pi\mu_n^2}\left\{\left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-1} J_0(\mu_n) - 1\right\}.$$

7. Boundary condition III-I.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_0(k\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(k\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$[Y_0(k\mu_n) J_0(\mu_n) - J_0(k\mu_n) Y_0(\mu_n)] \mathrm{Bi}_1 + [Y_0(k\mu_n) J_1(\mu_n) -$$
$$- J_0(k\mu_n) Y_1(\mu_n)] \mu_n = 0; \ \mathrm{Bi}_1 = h_1 R_1;$$
$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \left\{ 1 - \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n) \right]^{-2} J_0^2(k\mu_n) \left(1 + \frac{\mu_n^2}{\mathrm{Bi}_1^2} \right) \right\};$$
$$N_n = \frac{2R_1^2}{\pi \mu_n^2} \left\{ 1 - \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n) \right]^{-1} J_0(k\mu_n) \right\}.$$

8. Boundary condition II-III.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$[Y_1(\mu_n) J_0(k\mu_n) - J_1(\mu_n) Y_0(k\mu_n)] \mathrm{Bi}_2 - [Y_1(\mu_n) J_1(k\mu_n) -$$
$$- J_1(\mu_n) Y_1(k\mu_n)] \mu_n = 0; \ \mathrm{Bi}_2 = h_2 R_1;$$
$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \left\{ \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n) \right]^{-2} J_1^2(\mu_n) \left(1 + \frac{\mu_n^2}{\mathrm{Bi}_2^2} \right) - 1 \right\};$$
$$N_n = \frac{2R_1^2}{\pi \mu_n^2} \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n) \right]^{-1} J_1(\mu_n).$$

9. Boundary condition III-II.

Eigenfunction

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(k\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(k\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right).$$

Characteristic equation

$$\mathrm{Bi}_1\left[Y_1(k\mu_n)J_0(\mu_n)-J_1(k\mu_n)Y_0(\mu_n)\right]+\mu_n\left[Y_1(k\mu_n)J_1(\mu_n)-J_1(k\mu_n)Y_1(\mu_n)\right]=0;$$

$$\|U_0\|^2=\frac{2R_1^2}{\pi^2\mu_n^2}\left\{1-\left[J_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}J_1(\mu_n)\right]^{-2}J_1^2(k\mu_n)\left(1+\frac{\mu_n^2}{\mathrm{Bi}_1^2}\right)\right\};$$

$$N_n=-\frac{2R_1^2}{\pi\mu_n^2}\left[J_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}J_1(\mu_n)\right]^{-1}J_1(k\mu_n);\quad \mathrm{Bi}_1=h_1R_1.$$

Supplementary F Functions of the First Kind

The supplementary F functions of the first kind $F_I(\xi)$ satisfy the differential equation

$$\nabla^2 F_I=0$$

with boundary conditions of the first, second or third kind or

$$\nabla^2 F_I=\mathrm{const}$$

with boundary conditions of the second kind at both ends of the interval $[R_1, R_2]$.

Depending on whether function $F_I(\xi)$ satisfies the specific boundary condition "a" or "b" (see Tables 3-1 and 3-2), the letter "a" or "b" is added to the subscript.

As before

$$\bar{F}_I=\int_{R_1}^{R_2}\xi^i F_I(\xi)\,U_0\left(\mu_n\frac{\xi}{R}\right)d\xi\quad (i=0\vee 1).$$

In accordance with the data of Tables 3-1 and 3-2, the F functions are used to construct the substitution function $\Phi(\xi, \tau)$.

1. Boundary condition I-I.

Wall

$$F_{Ia}=\frac{x}{R};\quad \bar{F}_{Ia}=\frac{(-1)^{n+1}R}{\mu_n};$$

$$F_{Ib}=1-\frac{x}{R};\quad \bar{F}_{Ib}=\frac{R}{\mu_n}.$$

Hollow cylinder

$$F_{\mathrm{I}a} = \frac{1}{\ln k}\ln\frac{r}{R_1};\quad \bar{F}_{\mathrm{I}a} = \frac{2R_1^2}{\pi\mu_n^2}\,\frac{J_0(\mu_n)}{J_0(k\mu_n)};\quad k = \frac{R_2}{R_1};$$

$$F_{\mathrm{I}\text{б}} = 1 - \frac{1}{\ln k}\ln\frac{r}{R_1};\quad \bar{F}_{\mathrm{I}\text{б}} = -\frac{2R_1^2}{\pi\mu_n^2}.$$

2. Boundary condition II-II.

Wall

$$F_{\mathrm{I}a} = \frac{R}{2\lambda}\left(\frac{x^2}{R^2} - \frac{1}{3}\right);\quad \bar{F}_{\mathrm{I}a} = \frac{(-1)^n R^2}{\lambda\mu_n^2};$$

$$F_{\mathrm{I}\text{б}} = \frac{R}{2\lambda}\left(\frac{x^2}{R^2} - 2\frac{x}{R} + \frac{2}{3}\right);\quad \bar{F}_{\mathrm{I}\text{б}} = \frac{R^2}{\lambda\mu_n^2}.$$

Hollow cylinder

$$F_{\mathrm{I}a} = \frac{kR_1}{2\lambda(k^2-1)}\left(\frac{r^2}{R_1^2} - 2\ln\frac{r}{R_1} + \frac{2k^2\ln k}{k^2-1} - \frac{k^2+3}{2}\right);$$

$$\bar{F}_{\mathrm{I}a} = -\frac{2R_1^3}{\pi\lambda\mu_n^3}\,\frac{J_1(\mu_n)}{J_1(k\mu_n)};$$

$$F_{\mathrm{I}\text{б}} = \frac{R_1}{2\lambda(k^2-1)}\left(\frac{r^2}{R_1^2} - 2k^2\ln\frac{r}{R_1} + \frac{2k^4\ln k}{k^2-1} - \frac{k^2+3}{k^2-1}\right);$$

$$\bar{F}_{\mathrm{I}\text{б}} = -\frac{2R_1^3}{\pi\lambda\mu_n^3}.$$

3. Boundary condition III-III.

Wall

$$F_{\text{I}a} = \frac{\text{Bi}_2}{\text{Bi}_1 + \text{Bi}_2 + \text{Bi}_1\text{Bi}_2}\left(1 + \text{Bi}_1\frac{x}{R}\right); \quad \text{Bi}_1 = h_1R; \quad \text{Bi}_2 = h_2R;$$

$$\bar{F}_{\text{I}a} = (-1)^{n+1}\frac{\text{Bi}_2 R}{\mu_n}\sqrt{\frac{\mu_n^2 + \text{Bi}_1^2}{\mu_n^2 + \text{Bi}_2^2}};$$

$$F_{\text{Iб}} = \frac{\text{Bi}_1}{\text{Bi}_1 + \text{Bi}_2 + \text{Bi}_1\text{Bi}_2}\left(1 + \text{Bi}_2 - \text{Bi}_2\frac{x}{R}\right); \quad \bar{F}_{\text{Iб}} = \frac{\text{Bi}_1 R}{\mu_n}.$$

Hollow cylinder

$$F_{\text{I}a} = \frac{k\,\text{Bi}_1\,\text{Bi}_2}{\text{Bi}_1 + \text{Bi}_2 + \text{Bi}_1\text{Bi}_2\ln k}\left(\frac{1}{\text{Bi}_1} + \ln\frac{r}{R_1}\right); \quad \text{Bi}_1 = h_1R_1; \quad \text{Bi}_2 = h_2R_1;$$

$$\bar{F}_{\text{I}a} = \frac{2R_1^2\,\text{Bi}_2}{\pi\mu_n^2\,\text{Bi}_1}\,\frac{J_0(\mu_n) + \frac{\mu_n}{\text{Bi}_1}J_1(\mu_n)}{J_0(k\mu_n) - \frac{\mu_n}{\text{Bi}_2}J_1(k\mu_n)};$$

$$F_{\text{Iб}} = \frac{\text{Bi}_1}{\text{Bi}_1 + k\,\text{Bi}_2 + k\text{Bi}_1\text{Bi}_2\ln k}\left(-k\,\text{Bi}_2\ln\frac{r}{R_1} + 1 + k\text{Bi}_2\ln k\right);$$

$$\bar{F}_{\text{Iб}} = -\frac{2R_1^2}{\pi\mu_n^2}.$$

4. Boundary condition I-II.

Wall

$$F_{\text{I}a} = \frac{R}{\lambda}\frac{x}{R}; \quad \bar{F}_{\text{I}a} = \frac{(-1)^n R^2}{\lambda\mu_n^2}.$$

Hollow cylinder

$$F_{\text{I}a} = \frac{kR_1}{\lambda}\ln\frac{r}{R_1}; \quad \bar{F}_{\text{I}a} = \frac{2R_1^3}{\pi\lambda\mu_n^3}\frac{J_0(\mu_n)}{J_1(k\mu_n)}.$$

5. Boundary condition II-I.

Wall

$$F_{I\delta} = \frac{R}{\lambda}\left(1 - \frac{x}{R}\right); \quad \bar{F}_{I\delta} = \frac{R^2}{\lambda\mu_n^2}.$$

Hollow cylinder

$$F_{I\delta} = \frac{R_1}{\lambda}\left(\ln k - \ln\frac{r}{R_1}\right); \quad \bar{F}_{I\delta} = -\frac{2R_1^3}{\pi\lambda\mu_n^3}.$$

6. Boundary condition I-III.

Wall

$$F_{Ia} = \frac{\mathrm{Bi}_2}{1+\mathrm{Bi}_2}\frac{x}{R}; \quad \bar{F}_{Ia} = \frac{(-1)^{n+1}\mathrm{Bi}_2 R}{\mu_n\sqrt{\mu_n^2+\mathrm{Bi}_2^2}};$$

$$F_{I\delta} = -\frac{\mathrm{Bi}_2}{1+\mathrm{Bi}_2}\frac{x}{R}; \quad \bar{F}_{I\delta} = \frac{R}{\mu_n}.$$

Hollow cylinder

$$F_{Ia} = \frac{k\,\mathrm{Bi}_2}{(1+k\,\mathrm{Bi}_2\ln k)}; \quad \bar{F}_{Ia} = \frac{2R_1^2}{\pi\mu_n}\frac{J_0(\mu_n)}{J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2}J_1(k\mu_n)};$$

$$F_{I\delta} = 1 - \frac{k\,\mathrm{Bi}_2}{1+k\,\mathrm{Bi}_2\ln k}\ln\frac{r}{R_1}; \quad \bar{F}_{I\delta} = -\frac{2R_1^2}{\pi\mu_n^2}.$$

7. Boundary condition III-I.

Wall

$$F_{Ia} = \frac{1}{1+\mathrm{Bi}_1}\left(1+\mathrm{Bi}_1\frac{x}{R}\right); \quad \bar{F}_{Ia} = \frac{(-1)^{n+1}R}{\mu_n}\sqrt{\mu_n^2+\mathrm{Bi}_1^2};$$

$$F_{I\delta} = \frac{\mathrm{Bi}_1}{1+\mathrm{Bi}_1}\left(1-\frac{x}{R}\right); \quad \bar{F}_{I\delta} = \frac{R\,\mathrm{Bi}_1}{\mu_n}.$$

Hollow cylinder

$$F_{Ia} = \frac{1}{1 + \mathrm{Bi}_1 \ln k}\left(1 + \mathrm{Bi}_1 \ln \frac{r}{R_1}\right); \quad \bar{F}_{Ia} = \frac{2R_1^2}{\pi\mu_n^2};$$

$$F_{I6} = \frac{\mathrm{Bi}_1}{1 + \mathrm{Bi}_1 \ln k}\left(\ln k - \ln \frac{r}{R_1}\right); \quad \bar{F}_{I6} = \frac{2R_1^2}{\pi\mu_n^2} \frac{J_0(k\mu_n)}{J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)}.$$

8. Boundary condition II-III.

Wall

$$F_{I6} = \frac{R}{\lambda}\left(1 + \frac{1}{\mathrm{Bi}_2} - \frac{x}{R}\right); \quad \bar{F}_{I6} = \frac{R^2}{\lambda\mu_n^2}.$$

Hollow cylinder

$$F_{I6} = \frac{R_1}{\lambda}\left(\ln k + \frac{1}{k\,\mathrm{Bi}_2} - \ln \frac{r}{R_1}\right); \quad \bar{F}^{I6} = -\frac{2R_1^3}{\pi\lambda\mu_n^3}.$$

9. Boundary condition III-II.

Wall

$$F_{Ia} = \frac{R}{\lambda}\left(\frac{1}{\mathrm{Bi}_1} + \frac{x}{R}\right); \quad \bar{F}_{Ia} = \frac{(-1)^{n+1}R^2\sqrt{\mu_n^2 + \mathrm{Bi}_1^2}}{\lambda\mu_n^2}.$$

Hollow cylinder

$$F_{Ia} = \frac{kR_1}{\lambda}\left(\frac{1}{\mathrm{Bi}_1} + \ln \frac{r}{R_1}\right); \quad \bar{F}_{Ia} = -\frac{2R_1^3}{\pi\lambda\mu_n^3}.$$

Wall (0 < x < R). Finite-Difference Solutions

In the wall in the process of curing of the concrete, heat is liberated, the intensity of heat liberation depending on the temperature and time according to the relationships of I. D. Zaporozhets (see § 2-2). The initial temperature is a function of the coordinate. The boundary conditions are of the third kind, ambient temperature is a function of time.

$$\frac{\partial T}{\partial \tau}=a\frac{\partial^2 T}{\partial x^2}+\frac{1}{c\gamma}q(\tau, T)\ (0<x<R, 0<\tau<\theta); \tag{4-13}$$

$$T(x, 0)=f(x)\ (0\leqslant x\leqslant R); \tag{4-14}$$

$$\frac{\partial T(0, \tau)}{\partial x}=-\beta_1[\psi_1(\tau)-T(0, \tau)]; \tag{4-15}$$

$$\frac{\partial T(R, \tau)}{\partial x}=\beta_2[\psi_2(\tau)-T(R, \tau)].$$

Here

$$q(\tau, T)=q_0 2^{\frac{T-20}{\epsilon}}\left[1+A_{20}\int_0^{\tau}2^{\frac{T-20}{\epsilon}}d\tau\right]^{-s};\ q_0=\frac{Q_{max}A_{20}}{m-1};\ s=\frac{m}{m-1}. \tag{4-16}$$

This problem is nonlinear. It can only be solved by numerical methods. Let us use the method of finite differences.

We introduce the space-time grid $\omega_{R\ell}=\{x_i=ih,\ \tau_k=k\ell;\ i=0, 1, 2, \ldots, n;\ k=0, 2, \ldots, m\}$ and the grid function $T_{i,k}$. Let us approximate the differential equation with finite-difference relationships, using an implicit absolutely stable plan with lead (see § 3-6), and approximate the boundary conditions of the third kind using formulas presented in § 3-1, following from the equations of thermal balance on the surface of the body.

We have the difference problem

$$MT_{i+1, k+1}-(1+2M)T_{i, k+1}+MT_{i-1, k+1}=$$

$$=-\vartheta^{(k)}\ (1\leqslant i\leqslant n-1, 0\leqslant k\leqslant m-1); \tag{4-17}$$

$$T_{i, 0}=f(x_i)\ (0\leqslant i\leqslant n); \tag{4-18}$$

$$T_{0, k+1}=\varkappa_1 T_{1, k+1}+\theta_1^{(k)};$$

$$T_{n, k+1}=\varkappa_2 T_{n-1, k+1}+\theta_2^{(k)}, \tag{4-19}$$

where

$$\vartheta^{(k)} = T_{i,k} + \frac{q_0 l}{c\gamma} 2^{\frac{T_{i,k}-20}{s}} \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{i,p}-20}{s}}\right]^{-s};$$

$$\varkappa_j = \frac{M}{0{,}5 + M + MN_j};\quad N_j = \beta_j h = \frac{\alpha_j h}{\lambda}\ (j = 1, 2);\quad M = \frac{al}{h^2};$$

$$\theta_1^{(k)} = \frac{\varkappa_1}{M}\left\{0{,}5T_{0,k} + N_1 M \psi_{1,k+\frac{1}{2}} + \frac{q_0 l}{2c\gamma} 2^{\frac{T_{0,k}-20}{s}} \times \right.$$
$$\left. \times \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{0,p}-20}{s}}\right]^{-s}\right\};$$

$$\theta_2^{(k)} = \frac{\varkappa_2}{M}\left\{0{,}5T_{n,k} - N_2 M \psi_{2,k+\frac{1}{2}} + \frac{q_0 l}{2c\gamma} 2^{\frac{T_{n,k}-20}{s}} \times \right.$$
$$\left. \times \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{n,p}-20}{s}}\right]^{-s}\right\}.$$

Here, using some arbitrariness in the selection of the moment in time (τ_k, $\tau_{k+1/2}$ or τ_{k+1}) for heterogeneous terms of the differential equation and boundary conditions, we have accepted as the function of intensity of heat liberation moment in time τ_k [placing in formula (4-16) $T \to T_{i,k}$], for the temperature of the medium -- moment in time $\tau_{k+1/2}$ (assuming $\psi_j(\tau) \to$ $\to \psi_{j,\ k+1/2}$, $j = 1, 2$).

The system of algebraic equations (4-17) with boundary conditions (4-18) and (4-19) will be solved by the run-through method [42, 110].

In accordance with the theory of the method, we represent the grid function $T_{i,k+1}$ as:

$$T_{i,k+1} = \nu_{i+1} T_{i+1,k+1} + \varepsilon_{i+1}\ (i = 0, 1, \ldots, n-1), \tag{4-20}$$

where ν_{i+1}, ε_{i+1} are the coefficients to be determined.

It follows from this relationship that

$$T_{i-1,k+1}=\nu_i T_{i,k+1}+\varepsilon_i=\nu_i\nu_{i+1}T_{i+1,k+1}+\nu_i\varepsilon_{i+1}+\varepsilon_i.$$

Substitution of the last expressions into equation (4-17) yields:

$$[M-(1+2M-M\nu_i)\nu_{i+1}]T_{i+1,k+1}+[M\varepsilon_i+\vartheta^{(k)}-(1+2M-M\nu_i)\varepsilon_{i+1}]=0.$$

In order to satisfy this last equation, we require that

$$M-(1+2M-M\nu_i)\nu_{i+1}=0; \qquad M\varepsilon_i+\vartheta^{(k)}-(1+2M-M\nu_i)\varepsilon_{i+1}=0.$$

From this we produce:

$$\nu_{i+1}=\frac{M}{1+2M-M\nu_i};$$

$$\varepsilon_{i+1}=\frac{M\varepsilon_i+\vartheta^{(k)}}{1+2M-M\nu_i}. \tag{4-21}$$

Where i = 0, relationship (4-20) becomes:

$$T_{0,k+1}=\nu_1 T_{1,k+1}+\varepsilon_1.$$

Comparing this expression with the first boundary condition from (4-19), we find:

$$\nu_1=\varkappa_1;\ \varepsilon_1=\vartheta_1^{(k)}.$$

Consequently, using recurrent formulas (4-21) by "direct movement" (from $i = 1$ to $i = n - 1$) we establish the values of the run-through coefficients ν_{i+1} and ε_{i+1}.

The grid function $T_{i,k+1}$ is determined using formula (4-20) in "reverse gear" (from $i = n - 1$ to $i = 0$). The value of the grid function at the right boundary $T_{n,k+1}$ necessary to do this is produced from formula (4-20) where $i = n - 1$ and the second boundary condition from (4-19). We have:

$$T_{n,k+1}=\frac{\theta_2^{(k)}+\varkappa_2\varepsilon_n}{1-\varkappa_2\nu_n}.$$

Thus, the solution of the edge problem as stated can be represented in the form of the algorithm

$$\nu_{i+1}=\frac{M}{1+2M-M\nu_i};\ \varepsilon_{i+1}=\frac{M\varepsilon_i+\theta_i^{(k)}}{1+2M-M\nu_i}\ (i=1,2,\ldots,n-1);$$
$$\nu_1=\varkappa_1;\ \varepsilon_1=\theta_1^{(k)};$$
$$T_{i,k+1}=\nu_{i+1}T_{i+1,k+1}+\varepsilon_{i+1}\ (i=0,1,\ldots,n-1);$$
$$T_{n,k+1}=\frac{\theta_2^{(k)}+\varkappa_2\varepsilon_n}{1-\varkappa_2\nu_n}. \qquad (4\text{-}22)$$

This algorithm allows us to calculate the temperature function $T_{i,k}$ from layer to layer, beginning with the initial layer (where k = 0), defined by the assigned initial condition (4-18).

Let us write equation (4-17) in the form

$$A_iT_{i-1,k+1}-C_iT_{i,k+1}+B_iT_{i+1,k+1}=-\theta_i^{(k)}.$$

It is known from the theory [110], that the run-through algorithm (4-22) is stable, i.e., in the process of calculation the rounding errors do not increase if

$$A_i>0,\ B_i>0,\ C_i\geq A_i+B_i,\ 0\leq\varkappa_j<1\ (j=1,2).$$

In the case here in question

$$A_i=B_i=M>0;$$
$$C_i=1+2M>A_i+B_i=2M;$$
$$\varkappa_j=\frac{M}{0.5+M+MN_j},\ \text{т. е. } 0\leq\varkappa_j<1\ (j=1,2).$$

Consequently, the conditions of stability of run-through (4-22) are fulfilled.

The implicit plan (4-17), (4-19) has accuracy $O(h^2 + \ell)$. In order to increase the accuracy to $O(h^2 + \ell^2)$, the following sequence of solution is recommended (this plan is sometimes called a predictor-corrector).

At first we use an implicit plan with lead, with step 0.5 ℓ and with the heat liberation intensity function at moment τ_k. This gives us:

$$
\begin{gathered}
0{,}5MT_{i+1,\,k+1/2}-(1+M)\,T_{i,\,k+1/2}+0{,}5MT_{i-1,\,k+1/2}=-\vartheta^{(k)};\\
T_{i,\,0}=f(x_i);\\
T_{0,\,k+1/2}=\varkappa_1 T_{1,\,k+1/2}+\theta_1^{(k)};\\
T_{n,\,k+1/2}=\varkappa_2 T_{n-1,\,k+1/2}+\theta_2^{(k)},
\end{gathered}
\tag{4-23}
$$

where

$$
\vartheta^{(k)}=T_{i,\,k}+\frac{q_0 l}{2c\gamma}\,2^{\frac{T_{i,\,k}-20}{\varepsilon}}\left[1+A_{20}l\sum_{p=0}^{k}2^{\frac{T_{i,\,p}-20}{\varepsilon}}\right]^{-\varepsilon};
$$

$$
\varkappa_j=\frac{M}{1+M+MN_j};\quad N_j=\beta_j h\,(j=1,2);\quad M=\frac{al}{2h^2};
$$

$$
\theta_1^{(k)}=\frac{\varkappa_1}{M}\left\{T_{0,\,k}+MN_1\psi_{1,\,k+1/2}+\frac{q_0 l}{2c\gamma}\,2^{\frac{T_{0,\,k}-20}{\varepsilon}}\times\right.
$$

$$
\left.\times\left[1+A_{20}l\sum_{p=0}^{k}2^{\frac{T_{0,\,p}-20}{\varepsilon}}\right]^{-\varepsilon}\right\};
$$

$$
\theta_2^{(k)}=\frac{\varkappa_2}{M}\left\{T_{n,\,k}-MN_2\psi_{2,\,k+1/2}+\frac{q_0 l}{2c\gamma}\,2^{\frac{T_{n,\,k}-20}{\varepsilon}}\times\right.
$$

$$
\left.\times\left[1+A_{20}l\sum_{p=0}^{k}2^{\frac{T_{n,\,p}-20}{\varepsilon}}\right]^{-\varepsilon}\right\}.
$$

Solving problem (4-23) by the run-through method, we produce the intermediate value of temperature $T_{i,k+1/2}$.

We then apply with step ℓ the implicit symmetrical six-point Crank-Nicholson plan. As was noted in § 3-6, this plan can be produced from a single-parameter set of difference plans and boundary conditions if we assume $\sigma = 0.5$.

We have the difference problem

$$0{,}5MT_{i-1,k+1} - (1+M)T_{i,k+1} + 0{,}5MT_{i+1,k+1} = -\vartheta^{(k)};$$

$$T_{i,0} = f(x_i);$$

$$T_{0,k+1} = \varkappa_1 T_{1,k+1} + \theta_1^{(k)}; \quad T_{n,k+1} = \varkappa_2 T_{n-1,k+1} + \theta_2^{(k)}, \qquad (4\text{-}24)$$

where

$$\vartheta^{(k)} = (1-M)T_{i,k} + 0{,}5M(T_{i-1,k} + T_{i+1,k}) + \frac{q_0 l}{c\gamma} 2^{\frac{T_{i,k+1/2}-20}{\varepsilon}} \times$$

$$\times \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{i,p}-20}{\varepsilon}} + A_{20}\frac{l}{2} 2^{\frac{T_{i,k+1/2}-20}{\varepsilon}}\right]^{-\varepsilon};$$

$$\varkappa_j = \frac{M}{1 + M + MN_j}; \quad N_j = \beta_j h \ (j = 1,2);$$

$$\theta_1^{(k)} = \varkappa_1 \Big\{ T_{1,k} + \frac{1}{M}(1 - M - MN_1) T_{0,k} + 2N_1 \psi_{1,k+1/2} +$$

$$+ \frac{q_0 l}{Mc\gamma} 2^{\frac{T_{0,k+1/2}-20}{\varepsilon}} \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{0,p}-20}{\varepsilon}} + A_{20}\frac{l}{2} 2^{\frac{T_{0,k+1/2}-20}{\varepsilon}}\right]^{-\varepsilon} \Big\};$$

$$\theta_2^{(k)} = \varkappa_2 \Big\{ T_{n-1,k} + \frac{1}{M}(1 - M - MN_2) T_{n,k} - 2N_2 \psi_{2,k+1/2} +$$

$$+ \frac{q_0 l}{Mc\gamma} 2^{\frac{T_{n,k+1/2}-20}{\varepsilon}} \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{n,p}-20}{\varepsilon}} + A_{20}\frac{l}{2} 2^{\frac{T_{n,k+1/2}-20}{\varepsilon}}\right]^{-\varepsilon} \Big\}.$$

The solution of problem (4-24) by the run-through method yields the algorithm

$$\nu_{i+1} = \frac{0{,}5M}{1 + M - 0{,}5M\nu_i}; \quad \varepsilon_{i+1} = \frac{0{,}5M\varepsilon_i + \vartheta^{(k)}}{1 + M - 0{,}5M\nu_i} \ (i = 1, \ldots, n-1);$$

$$\nu_1 = \varkappa_1; \quad \varepsilon_1 = \theta_1^{(k)};$$

$$T_{i,k+1} = \nu_{i+1} T_{i+1,k+1} + \varepsilon_{i+1} \ (i = 0, 1, \ldots, n-1);$$

$$T_{n,k+1} = \frac{\theta_2^{(k)} + \varkappa_2 \varepsilon_n}{1 - \varkappa_2 \nu_n}.$$

As before, all of the conditions required for run-through stability are fulfilled here.

4-3. Calculations of Temperature Fields of Structural Elements Using Two-Dimensional Plans

Temperature fields which are near two-dimensional are observed throughout a significant portion of the volume of massive concrete hydraulic engineering structures. Since it is impossible to present algorithms for all possible versions of boundary conditions, let us discuss but a few of these, paying particular attention to presentation of the method of solution of the corresponding problems.

Half strip ($0 < x < R$, $0 < y < \infty$) and semilimited cylinder ($0 < r < R$ or $R_1 < r < R_2$, $0 < z < \infty$)

1. The initial distribution of temperature in the body is represented as a derivative of two functions, each of which depends only on one coordinate. The temperature of the medium (boundary condition of third kind) or surface (boundary condition of first kind) is constant.

For definition, let us study the problem for the half strip ($0 < x < R$, $0 < y < \infty$) and semilimited hollow cylinder ($R_1 < r < R_2$, $0 < z < \infty$) with boundary conditions of the third kind at the end surface $\zeta = 0$ and side surface $\xi = R_2$ and with conditions of the first kind on the side surface $\xi = R_1$.

The differential equation

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right) + \frac{\partial^2 T}{\partial \zeta^2}\right] \times$$
$$\times (R_1 < \xi < R_2, 0 < \zeta < \infty, \tau > 0, i = 0 \vee 1)^1. \qquad (4\text{-}25)$$

The initial condition

$$T(\xi,\zeta,0) = f_1(\xi) f_2(\zeta) \quad (R_1 \leqslant \xi \leqslant R_2, 0 \leqslant \zeta < \infty). \qquad (4\text{-}26)$$

The boundary condition

$$T(R_1,\zeta,\tau) = T_1;$$
$$\frac{\partial T(R_2,\zeta,\tau)}{\partial \xi} = h_2[T_1 - T(R_2,\zeta,\tau)];$$
$$\frac{\partial T(\xi,0,\tau)}{\partial \zeta} = -h_3[T_1 - T(\xi,0,\tau)];$$
$$\frac{\partial T(\xi,\infty,\tau)}{\partial \zeta} = 0, \quad T(\xi,\infty,\tau) \neq \infty. \qquad (4\text{-}27)$$

[1] For the half strip: $\xi = x$, $\zeta = y$, $i = 0$, $R_1 = 0$, $R_2 = R$; for the semilimited hollow cylinder: $\xi = r$, $\zeta = z$, $i = 1$.

Let us assume

$$T = T_1 - u - v,$$

where the function $u(\xi, \zeta, \tau)$ satisfies equation (4-25), initial condition

$$u(\xi, \zeta, 0) = T_1$$

and the homogeneous boundary conditions

$$\begin{gathered} u(R_1, \zeta, \tau) = 0; \\ \frac{\partial u(R_2, \zeta, \tau)}{\partial \xi} = -h_2 u(R_2, \zeta, \tau); \\ \frac{\partial u(\xi, 0, \tau)}{\partial \zeta} = h_3 u(\xi, 0, \tau); \\ \frac{\partial u(\xi, \infty, \tau)}{\partial \zeta} = 0; \;\; u(\xi, \infty, \tau) \neq \infty, \end{gathered} \tag{4-27'}$$

while function $v(\xi, \zeta, \tau)$ satisfies equation (4-25), the initial condition

$$v(\xi, \zeta, 0) = -f_1(\xi) f_2(\zeta)$$

and boundary conditions (4-27'), but homogeneous.

Based on the property of multiplication of solutions (see § 3-1), we can write for functions u and v:

$$\begin{gathered} u = \theta_1(\xi, \tau)\theta_2(\zeta, \tau); \\ v = \theta_3(\xi, \tau)\theta_4(\zeta, \tau), \end{gathered}$$

where function θ_j (j = 1, 2, 3, 4) are solutions of the one-dimensional problems, namely: function $\theta_1(\xi, \tau)$ satisfies the equation

$$\frac{\partial \theta_1}{\partial \tau} = a \frac{1}{\xi^i} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial \theta_1}{\partial \xi}\right) \; (R_1 < \xi < R_2, \tau > 0; \; i = 0 \vee 1), \tag{4-28}$$

the initial condition

$$\theta_1(\xi,0)=T_1$$

and the homogeneous boundary conditions

$$\theta_1(R_1,\tau)=0;$$
$$\frac{\partial\theta_1(R_2,\tau)}{\partial\xi}=-h_2\theta_1(R_2,\tau). \qquad (4\text{-}29)$$

Function $\theta_3(\xi,\tau)$ satisfies equation (4-28), boundary conditions (4-29) and the initial condition

$$\theta_3(\xi,0)=-f_1(\xi);$$

function $\theta_2(\zeta,\tau)$ satisfies equation

$$\frac{\partial\theta_2}{\partial\tau}=a\frac{\partial^2\theta_2}{\partial\zeta^2}\,(0<\zeta<\infty,\tau>0), \qquad (4\text{-}30)$$

initial condition

$$\theta_2(\zeta,0)=1$$

and the homogeneous boundary conditions

$$\frac{\partial\theta_2(0,\tau)}{\partial\zeta}=h_3\theta_2(0,\tau);\ \frac{\partial\theta_2(\infty,\tau)}{\partial\zeta}=0;\ \theta_2(\infty,\tau)\neq\infty. \qquad (4\text{-}31)$$

Function $\theta_4(\zeta,\tau)$ satisfies equation (4-30), boundary conditions (4-31) and the initial condition

$$\theta_4(\zeta,0)=-f_2(\zeta).$$

The solution of all these problems can be easily written.

We have:

$$\theta_1(\xi,\tau)=T_1\sum_{n=1}^{\infty}A_nU_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}};$$

$$\theta_3(\xi,\tau)=-\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\int_{R_1}^{R_2}\xi^i f_1(\xi)\,U_0\left(\mu_n\frac{\xi}{R}\right)d\xi\,U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}};$$

$$\theta_2(\zeta,\tau)=1-\left(\operatorname{erfc}\left[\frac{\zeta}{2\sqrt{a\tau}}\right]-\exp\left[h_3^2a\tau+h_3\zeta\right]\times\right.$$

$$\left.\times\operatorname{erfc}\left[h_3\sqrt{a\tau}+\frac{\zeta}{2\sqrt{a\tau}}\right]\right);$$

$$\theta_4(\zeta,\tau)=-\int_0^{\infty}\left\{\frac{1}{2\sqrt{\pi a\tau}}\left(\exp\left[-\frac{(\zeta-\zeta_0)^2}{4a\tau}\right]+\exp\left[-\frac{(\zeta+\zeta_0)^2}{4a\tau}\right]\right)-\right.$$

$$\left.-h_3\exp\left[h_3^2a\tau+h_3(\zeta+\zeta_0)\right]\operatorname{erfc}\left[h_3\sqrt{a\tau}+\frac{\zeta+\zeta_0}{2\sqrt{a\tau}}\right]\right\}f_2(\zeta_0)\,d\zeta_0.$$

Here

$U_0(\mu_n\frac{\xi}{R})$ is the Eigenfunction of problem (4-28)-(4-29);

μ_n is the root of the characteristic equation;

$\|U_0\|^2=\int_{R_1}^{R_2}\xi^iU_0^2\left(\mu_n\frac{\xi}{R}\right)d\xi$ is the square of the norm of the Eigenfunction;

$$A_n=\frac{N_n}{\|U_0\|^2};\quad N_n=\int_{R_1}^{R_2}\xi^iU_0\left(\mu_n\frac{\xi}{R}\right)d\xi;$$

R is the characteristic dimension of the body (width of half strip R or radius of internal surface of hollow cylinder R_1).

From § 4-2, we find:

For a half strip

$$U_0\left(\mu_n\frac{\xi}{R}\right)=\sin\mu_n\frac{x}{R};$$

μ_n is the root of the characteristic equation

$$\tan\mu_n=-\frac{\mu_n}{\mathrm{Bi}_2};\quad \mathrm{Bi}_2=h_2R;$$

$$\|U_0\|^2=\frac{R}{2}\cdot\frac{\mathrm{Bi}_2^2+\mathrm{Bi}_2+\mu_n^2}{\mathrm{Bi}_2^2+\mu_n^2};\quad N_n=\frac{R}{\mu_n}\left[1+\frac{(-1)^{n+1}\mathrm{Bi}_2}{\sqrt{\mathrm{Bi}_2^2+\mu_n^2}}\right];\qquad (4\text{-}32)$$

For a semilimited hollow cylinder

$$U_0\left(\mu_n \frac{\xi}{R}\right) = Y_0(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right);$$

μ_n is the root of the characteristic equation

$$\mathrm{Bi}_2 [Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n)] -$$
$$- \mu_n [Y_0(\mu_n) J_1(k\mu_n) - J_0(\mu_n) Y_1(k\mu_n)] = 0; \quad k = \frac{R_2}{R_1};$$
$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \left(J_0^2(\mu_n) \left(1 + \frac{\mu_n^2}{\mathrm{Bi}_2^2}\right) \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n) \right]^{-2} - 1 \right);$$
$$N_n = \frac{2R_1^2}{\pi \mu_n^2} \left(J_0(\mu_n) \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n) \right]^{-1} - 1 \right).$$

Thus, the solution of problem (4-25)-(4-27) is:

$$T(\xi, \zeta, \tau) = T_1 - \theta_1(\xi, \tau)\theta_2(\zeta, \tau) - \theta_3(\xi, \tau)\theta_4(\zeta, \tau),$$

where the functions θ_j ($j = 1, 2, 3, 4$) are determined by the solutions presented above.

2. A semistrip ($0 < x < R$, $0 < y < \infty$). The initial temperature is constant T_0. The surface temperature $x = 0$ is equal to T_1, on the surfaces $x = R$ and $y = 0$ we fix boundary conditions of the third kind, and ambient temperatures T_2 and T_3 respectively.

The differential equation

$$\frac{\partial T}{\partial \tau} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (0 < x < R,\ 0 < y < \infty,\ \tau > 0). \tag{4-33}$$

The initial condition

$$T(x, y, 0) = T_0 \quad (0 \leq x \leq R,\ 0 \leq y < \infty).$$

The boundary conditions

$$T(0, y, \tau) = T_1;$$

$$\frac{\partial T(R, y, \tau)}{\partial x} = h_2 [T_2 - T(R, y, \tau)];$$

$$\frac{\partial T(x, 0, \tau)}{\partial y} = -h_3 [T_3 - T(x, 0, \tau)];$$

$$\frac{\partial T(x, \infty, \tau)}{\partial y} = 0, \; T(x, \infty, \tau) \neq \infty. \qquad (4\text{-}34)$$

The solution of problem (4-33)-(4-34) is produced by means of a finite integral transform with respect to x and the Green function with respect to y. The first pair of boundary conditions (4-34) is heterogeneous. Therefore, before beginning transformation with respect to x, we produce:

$$T(x, y, \tau) = \Phi(x) - \theta(x, y, \tau),$$

where $\Phi(x)$ is a substitution function, which we select as

$$\Phi(x) = T_1 + (T_2 - T_1) F_{Ia}(x),$$

$F_{Ia}(x)$ is a F function of the first kind, equal to

$$F_{Ia} = \frac{Bi_2}{1 + Bi_2} \frac{x}{R}; \quad Bi_2 = h_2 R$$

(see reference data; § 4-2).

Then, for function $\theta(x, y, \tau)$ we have:

the differential equation

$$\frac{\partial \theta}{\partial \tau} = a \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) (0 < x < R, \; 0 < y < \infty, \; \tau > 0);$$

initial condition

$$\theta(x, y, 0) = \Phi(x) - T_0 \; (0 \leqslant x \leqslant R, \; 0 \leqslant y < \infty);$$

boundary conditions

$$\theta(0, y, \tau) = 0;$$

$$\frac{\partial\theta(R, y, \tau)}{\partial x} = -h_2\theta(R, y, \tau);$$

$$\frac{\partial\theta(x, 0, \tau)}{\partial y} = -h_3[(\Phi(x) - T_3) - \theta(x, 0, \tau)];$$

$$\frac{\partial\theta(x, \infty, \tau)}{\partial y} = 0; \quad \theta(x, \infty, \tau) \neq \infty.$$

Applying a finite integral transform to the last problem, defined by the formulas:

$$\bar{\theta}_n(y, \tau) = \int_0^R \theta(x, y, \tau) U_0\left(\mu_n \frac{x}{R}\right) dx;$$

$$\theta(x, y, \tau) = \sum_{n=1}^{\infty} \frac{\bar{\theta}_n(y, \tau)}{\|U_0\|^2} U_0\left(\mu_n \frac{x}{R}\right).$$

The Eigenfunction $U_0(\mu_n \frac{x}{R})$, square of the norm $||U_0||^2$, the characteristic equation for μ_n, coefficient N_n has the same values as in the previous problem.

Using the standard method, we produce

$$\frac{\partial\bar{\theta}_n}{\partial\tau} = a\frac{\partial^2\bar{\theta}_n}{\partial y^2} - \frac{a\mu_n^2}{R^2}\bar{\theta}_n \quad (0 < y < \infty, \ \tau > 0);$$

$$\bar{\theta}_n'(y, 0) = \bar{\Phi}_n - T_0 N_n \quad (0 \leq y < \infty);$$

$$\frac{\partial\bar{\theta}_n(0, \tau)}{\partial y} = -h_3[(\bar{\Phi}_n - T_3 N_n) - \bar{\theta}_n(0, \tau)];$$

$$\frac{\partial\bar{\theta}_n(\infty, \tau)}{\partial y} = 0; \quad \bar{\theta}_n(\infty, \tau) \neq \infty,$$

where the transform of the substitution function $\bar{\Phi}_n$ is

$$\bar{\Phi}_n = T_1 N_n + (T_2 - T_1)\bar{F}_{in} = T_1 N_n + (T_2 - T_1)\frac{(-1)^{n+1}\mathrm{Bi}_2 R}{\mu_n\sqrt{\mathrm{Bi}_2^2 + \mu_n^2}}.$$

Performing the substitution

$$\bar{\vartheta}_n(y, \tau) = \bar{\vartheta}_n(y, \tau)\, e^{-\mu_n^2 \frac{a\tau}{R^2}}$$

yields:

$$\frac{\partial \bar{\vartheta}_n}{\partial \tau} = a \frac{\partial^2 \bar{\vartheta}_n}{\partial y^2} \quad (0 < y < \infty, \tau > 0);$$

$$\vartheta_n(y, 0) = \bar{\Phi}_n - T_0 N_n \quad (0 \le y < \infty);$$

$$\frac{\partial \bar{\vartheta}_n(0, \tau)}{\partial y} = -h_3 [(\bar{\Phi}_n - T_3 N_n)\, e^{\mu_n^2 \frac{a\tau}{R^2}} - \bar{\vartheta}_n(0, \tau)],$$

$$\frac{\partial \bar{\vartheta}_n(\infty, \tau)}{\partial y} = 0; \quad \bar{\vartheta}_n(\infty, \tau) \neq \infty.$$

To solve the problem for the transform $\bar{\theta}_n$, we use § 3-3.

We have:

$$\bar{\vartheta}_n = (\bar{\Phi}_n - T_0 N_n) \int_0^\infty \left\{ \frac{1}{2\sqrt{\pi a \tau}} \left(\exp\left[-\frac{(y - y_0)^2}{4a\tau} \right] + \right.\right.$$
$$+ \exp\left[-\frac{(y + y_0)^2}{4a\tau} \right] \Big) - h_3 \exp[h_3^2 a\tau + h_3 (y + y_0)] \operatorname{erfc}\left[\frac{y + y_0}{2\sqrt{a\tau}} + \right.$$
$$\left.\left. + h_3 \sqrt{a\tau} \right] \right\} dy_0 + h_3 a (\bar{\Phi}_n - T_3 N_n) \int_0^\tau \left\{ \frac{1}{\sqrt{\pi a (\tau - t)}} \times \right.$$
$$\times \exp\left[-\frac{y^2}{4a(\tau - t)} \right] - h_3 \exp[h_3^2 a(\tau - t) +$$
$$\left. + h_3 y] \operatorname{erfc}\left[\frac{y}{2\sqrt{a(\tau - t)}} + h_3 \sqrt{a(\tau - t)} \right] \right\} e^{\frac{\mu_n^2 a}{R^2} t} dt.$$

Let us take the integrals included in the last expression

$$1)\ I_1 = \int_0^\infty \left\{ \frac{1}{2\sqrt{\pi a \tau}} \left(\exp\left[-\frac{(y - y_0)^2}{4a\tau} \right] + \exp\left[-\frac{(y + y_0)^2}{4a\tau} \right] \right) - \right.$$
$$\left. - h_3 \exp[h_3^2 a\tau + h_3 (y + y_0)] \operatorname{erfc}\left[\frac{y + y_0}{2\sqrt{a\tau}} + h_3 \sqrt{a\tau} \right] \right\} dy_0.$$

We encountered a similar integral when we determined the temperature in a semilimited body with constant initial temperature.

Assuming in the corresponding formula $T_1 = 1$ and $T_c = 0$, we find:

$$I_1 = 1 - \left(\operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}}\right] - \exp\left[h_3^2 a\tau + h_3 y\right] \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} + h_3\sqrt{a\tau}\right]\right).$$

$$2)\ I_2 = e^{-\mu_n^2 \frac{a\tau}{R^2}} \int_0^\tau \frac{1}{\sqrt{\tau - t}} \exp\left[-\frac{y^2}{4a(\tau - t)}\right] \exp\left[\frac{\mu_n^2 at}{R^2}\right] dt =$$

$$= \int_0^\tau \frac{1}{\sqrt{t}} \exp\left[-\frac{\mu_n^2 a}{R^2} t - \frac{y^2}{4at}\right] dt.$$

We will show below how to select such integrals. The result

$$I_2 = \frac{R\sqrt{\pi}}{2\mu_n\sqrt{a}}\left(e^{-\frac{\mu_n}{R}y} \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} - \frac{\mu_n}{R}\sqrt{a\tau}\right] - \right.$$

$$\left. - e^{\frac{\mu_n}{R}y} \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} + \frac{\mu_n}{R}\sqrt{a\tau}\right]\right).$$

$$3)\ I_3 = e^{-\mu_n^2 \frac{a\tau}{R^2}} \int_0^\tau \exp\left[h_3^2 a(\tau - t) + h_3 y + \frac{\mu_n^2 at}{R^2}\right] \times$$

$$\times \operatorname{erfc}\left[\frac{y}{2\sqrt{a(\tau - t)}} + h_3\sqrt{a(\tau - t)}\right] dt =$$

$$= \int_0^\tau \exp\left[h_3^2 at - \frac{\mu_n^2 a}{R^2} t + h_3 y\right] \operatorname{erfc}\left[\frac{y}{2\sqrt{at}} + h_3\sqrt{at}\right] dt.$$

This integral is taken by parts.

The result

$$I_3 = \frac{R^2}{a(R^2 h_3^2 - \mu_n^2)} e^{-\mu_n^2 \frac{a\tau}{R^2}} \exp\left[h_3^2 a\tau + h_3 y\right] \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} + h_3\sqrt{a\tau}\right] +$$

$$+ \frac{h_3 R^2}{\sqrt{\pi a}(R^2 h_3^2 - \mu_n^2)} I_2 - \frac{R^2 y}{2\sqrt{\pi a}\, a (R^2 h_3^2 - \mu_n^2)} I_4,$$

where

$$I_1 = \int_0^{\tau} t^{-3/2} \exp\left[-\frac{\mu_n a}{R^2} t - \frac{y^2}{4at}\right] dt = \frac{\sqrt{\pi a}}{y}\left(e^{-\frac{\mu_n}{R} y} \times\right.$$

$$\left. \times \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} - \frac{\mu_n}{R}\sqrt{a\tau}\right] + e^{\frac{\mu_n}{R} y} \operatorname{erfc}\left[\frac{y}{2\sqrt{a\tau}} + \frac{\mu_n}{R}\sqrt{a\tau}\right]\right).$$

Consequently,

$$\bar{\theta}_n = \bar{\vartheta}_n e^{-\mu_n^2 \frac{a\tau}{R^2}} = (\bar{\Phi}_n - T_0 N_n) e^{-\mu_n^2 \frac{a\tau}{R^2}} I_1 +$$

$$+ h_3 \sqrt{\frac{a}{\pi}} (\bar{\Phi}_n - T_3 N_n) I_2 - h_3^2 a (\bar{\Phi}_n - T_3 N_n) I_3.$$

Based on the corresponding inversion formula, we have:

$$\theta(x, y, \tau) = \sum_{n=1}^{\infty} \frac{\bar{\theta}_n}{\| U_0 \|^2} U_0\left(\mu_n \frac{x}{R}\right).$$

From which we find

$$T(x, y, \tau) = \Phi(x) - \theta(x, y, \tau).$$

It is not difficult to continue all of the necessary computations to the end, and therefore we will not discuss them.

Rectangle

1. The rectangle ($-R < x < R$, $-L < y < L$). Initial temperature -- constant T_0. The faces of the rectangle are maintained at temperature T_1. Due to the symmetry of the temperature field, we place the coordinate origin at the center of the rectangle and analyze one quarter of the rectangle ($0 < x < R$, $0 < y < L$).

We introduce the function $\theta(x, y, \tau)$, defined by the formula

$$T = T_1 - \theta.$$

Then

$$\frac{\partial\theta}{\partial\tau} = a\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) \quad (0 < x < R,\ 0 < y < L,\ \tau > 0);$$
$$\theta(x, y, 0) = T_1 - T_0 \quad (0 \leq x \leq R,\ 0 \leq y \leq L);$$
$$\theta(R, y, \tau) = \theta(x, L, \tau) = 0; \qquad (4\text{-}35)$$
$$\frac{\partial T(0, y, \tau)}{\partial x} = \frac{\partial T(x, 0, \tau)}{\partial y} = 0 \quad \text{(condition of symmetry)}.$$

Using the property of multiplication of solutions, we can write:

$$\theta(x, y, \tau) = \theta_1(x, \tau)\,\theta_2(y, \tau),$$

where $\theta_1(x, \tau)$ and $\theta_2(y, \tau)$ are solutions of the one-dimensional problems, namely:

Function $\theta_1(x, \tau)$

$$\frac{\partial\theta_1}{\partial\tau} = a\frac{\partial^2\theta_1}{\partial x^2} \quad (0 < x < R,\ \tau > 0);$$
$$\theta_1(x, 0) = T_1 - T_0 \quad (0 \leq x \leq R);$$
$$\frac{\partial\theta_1(0, \tau)}{\partial x} = 0; \quad \theta_1(R, \tau) = 0; \qquad (4\text{-}36)$$

Function $\theta_2(y, \tau)$

$$\frac{\partial\theta_2}{\partial\tau} = a\frac{\partial^2\theta_2}{\partial y^2} \quad (0 < y < L,\ \tau > 0);$$
$$\theta_2(y, 0) = 1;$$
$$\frac{\partial\theta_2(0, \tau)}{\partial y} = 0; \quad \theta_2(L, \tau) = 0. \qquad (4\text{-}37)$$

The solutions of problems (4-36) and (4-37) are

$$\theta_1(x,\tau) = (T_1 - T_0)\sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}};$$

$$\theta_2(y,\tau) = \sum_{m=1}^{\infty} A_m V_0\left(\nu_m \frac{y}{L}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}},$$

where

$$U_0\left(\mu_n \frac{x}{R}\right) = \cos\mu_n \frac{x}{R};\quad \mu_n = (2n-1)\frac{\pi}{2}\ (n=1, 2\ \ldots);$$

$$V_0\left(\nu_m \frac{y}{L}\right) = \cos\nu_m \frac{y}{L};\quad \nu_m = (2m-1)\frac{\pi}{2}\ (m=1, 2\ \ldots);$$

$$A_n = \frac{4(-1)^{n+1}}{(2n-1)\pi};\qquad A_m = \frac{4(-1)^{m+1}}{(2m-1)\pi}.$$

Thus,

$$\frac{T(x,y,\tau) - T_1}{T_0 - T_1} = \left[\sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{x}{R}\right) e^{-\mu_n^2 \frac{a\tau}{R^2}}\right]\times$$
$$\times\left[\sum_{m=1}^{\infty} A_m V_0\left(\nu_m \frac{y}{L}\right) e^{-\nu_m^2 \frac{a\tau}{L^2}}\right].$$

2. Rectangle $(0 < x < R,\ 0 < y < L)$. Initial temperature -- constant T_0. One of the faces $(x = 0)$ is maintained at temperature T_1, while face $x = R$ has temperature T_2. The heat exchange of the face $y = L$ with the environment, the temperature of which is equal to 0, is by convection. Face $y = 0$ is insulated from heat.

The differential equation

$$\frac{\partial T}{\partial\tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\ (0 < x < R,\ 0 < y < L,\ \tau > 0).$$

The initial condition

$$T(x, y, 0) = T_0 \quad (0 \leqslant x \leqslant R,\ 0 \leqslant y \leqslant L). \tag{4-38}$$

The boundary condition

$$T(0, y, \tau) = T_1; \quad T(R, y, \tau) = T_2;$$
$$\frac{\partial T(x, 0, \tau)}{\partial y} = 0; \quad \frac{\partial T(x, L, \tau)}{\partial y} = -hT(x, L, \tau).$$

The problem will be solved with multiple finite integral transforms. First we perform the transform with respect to variable x. The formulas of this transform are

$$\bar{T}_n(y, \tau) = \int_0^R T(x, y, \tau) U_0\left(\mu_n \frac{x}{R}\right) dx;$$
$$T(x, y, \tau) = \sum_{n=1}^{\infty} \frac{\bar{T}_n(y, \tau)}{\| U_0 \|^2} U_0\left(\mu_n \frac{x}{R}\right), \tag{4-39}$$

where

$$U_0\left(\mu_n \frac{x}{R}\right) = \sin \mu_n \frac{x}{R}; \quad \mu_n = n\pi \quad (n = 1, 2 \ldots);$$
$$\| U_0 \|^2 = \frac{R}{2}; \quad N_n = \frac{2R}{(2n-1)\pi}.$$

We have

$$\frac{\partial \bar{T}_n}{\partial \tau} = a \frac{\partial^2 \bar{T}_n}{\partial y^2} - \frac{a\mu_n^2}{R^2} \bar{T}_n + (-1)^{n+1} \frac{a\mu_n}{R} T_2 + \frac{a\mu_n}{R} T_1;$$
$$\bar{T}_n(y, 0) = T_0 N_n;$$
$$\frac{\partial \bar{T}_n(0, \tau)}{\partial y} = 0; \quad \frac{\partial \bar{T}_n(L, \tau)}{\partial y} = -h\bar{T}_n(L, \tau). \tag{4-40}$$

The second transform with respect to variable y is defined by the formulas

$$\widetilde{T}_{nm}(\tau) = \int_0^L T_n(y, \tau) V_0\left(\varkappa_m \frac{y}{L}\right) dy;$$

$$T_n(y, \tau) = \sum_{m=1}^{\infty} \frac{\widetilde{T}_{nm}(\tau)}{\| V_0 \|^2} V_0\left(\varkappa_m \frac{y}{L}\right), \tag{4-41}$$

where $V_0(\kappa_m \frac{y}{L}) = \cos \kappa_m \frac{y}{L}$ is the Eigenfunction of problem (4-40); κ_m is the root of the characteristic equation

$$\cot \varkappa_m = \frac{\varkappa_m}{\mathrm{Bi}}; \quad \mathrm{Bi} = hL;$$

$$\| V_0 \|^2 = \frac{L}{2} \frac{\mathrm{Bi}^2 + \mathrm{Bi} + \varkappa_m^2}{\mathrm{Bi}^2 + \varkappa_m^2}.$$

The application of transform (4-41) to problem (4-40) yields:

$$\frac{d\widetilde{T}_{nm}}{d\tau} = -a\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)\widetilde{T}_{nm} + (-1)^{n+1} \frac{a\mu_n}{R} T_2 N_m + \frac{a\mu_n}{R} T_1 N_m;$$

$$\widetilde{T}_{nm}(0) = T_0 N_n N_m,$$

where

$$N_m = \frac{(-1)^{m+1} \mathrm{Bi}\, L}{\varkappa_m \sqrt{\mathrm{Bi}^2 + \varkappa_m^2}}.$$

From this

$$\widetilde{T}_{nm} = T_0 N_n N_m \exp\left[-\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right) a\tau\right] - \frac{\mu_n N_m}{R\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} \times$$

$$\times [(-1)^{n+1} T_2 + T_1] \exp\left[-\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right) a\tau\right] +$$

$$+ \frac{(-1)^{n+1} \mu_n N_m T_2}{R\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} + \frac{\mu_n N_m T_1}{R\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)}.$$

Successive application of the inversion formulas (4-39) and (4-41) leads to solution of the problem stated in the form

$$T = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_m}{(2n-1)} \sin \frac{n\pi x}{R} \cos \varkappa_m \frac{y}{L} \times$$
$$\times \exp\left[-\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)\right]$$
$$-\frac{2\pi}{R^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nA_m}{\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} [(-1)^{n+1}T_2 + T_1] \sin \frac{n\pi x}{R} \times$$
$$\times \cos \varkappa_m \frac{y}{L} \exp\left[-\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)\right] + T_2S_1 + T_1S_2, \tag{4-42}$$

where

$$S_1 = \frac{2\pi}{R^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+1} nA_m}{\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} \sin \frac{n\pi x}{R} \cos \varkappa_m \frac{y}{L}; \tag{4-43}$$

$$S_2 = \frac{2\pi}{R^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nA_m}{\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} \sin \frac{n\pi x}{R} \cos \varkappa_m \frac{y}{L}; \tag{4-44}$$

$$A_m = \frac{(-1)^{m+1} 2\mathrm{Bi}}{\varkappa_m} (\mathrm{Bi}^2 + \mathrm{Bi} + \varkappa_m^2)^{-1} \sqrt{\mathrm{Bi}^2 + \varkappa_m^2}.$$

Series (4-43) and (4-44) converge slightly with respect to n (as 1/n).

We present below several examples which allow us to improve the convergence of these series.

1. Series S_1.

Let us write series S_1 as

$$S_1 = \frac{2}{\pi} \sum_{m=1}^{\infty} A_m \cos \varkappa_m \frac{y}{L} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nz}{n^2 + p^2},$$

where

$$z = \frac{\pi x}{R}; \quad p^2 = \frac{\varkappa_m^2 R^2}{\pi^2 L^2}.$$

We know that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nz}{n^2 + p^2} = \frac{\pi}{2} \frac{\operatorname{sh} pz}{\operatorname{sh} p\pi} \quad (-\pi < z < \pi). \tag{4-45}$$

Therefore

$$S_1 = \sum_{m=1}^{\infty} A_m \frac{\operatorname{sh} \varkappa_m \frac{x}{L}}{\operatorname{sh} \varkappa_m \frac{R}{L}} \cos \varkappa_m \frac{y}{L}. \tag{4-43'}$$

Series (4-43') converges particularly well where R > L almost throughout the entire range of change of variables x and y, with the exception of points lying near the boundary x = R. Here the convergence of the series is of order $1/\kappa_m^2$ (due to the coefficients A_m).

We note that on the surface x = R, series (4-43'), due to limitations placed on sum (4-45), is not defined. Actually, there is no need for this, since the value of the temperature function where x = R is assigned by the boundary condition.

Series (4-43) can also be represented differently.

Let us write this series as

$$S_1 = \frac{2\pi}{R^2} \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \frac{n\pi x}{R} \sum_{m=1}^{\infty} \frac{A_m}{\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} \cos \varkappa_m \frac{y}{L}.$$

In order to sum the series with respect to m, let us analyze the solution of the following supplementary problem

$$\frac{d^2 w}{dy^2} - \frac{\beta^2}{L^2} w = -1 \quad (0 < y < L);$$

$$\frac{dw(0)}{dy} = 0; \qquad \frac{dw(L)}{dy} = -hw(L). \tag{4-46}$$

Applying integral transform (4-41) to problem (4-46), we find

$$-\frac{\varkappa_m^2}{L^2}\tilde{w}_m - \frac{\beta^2}{L^2}\tilde{w}_m = -N_m.$$

From this

$$\tilde{w}_m = \frac{L^2 N_m}{\varkappa_m^2 + \beta^2}$$

and on the basis of the inversion formula (4-41)

$$w = L^2 \sum_{m=1}^{\infty} \frac{A_m}{\varkappa_m^2 + \beta^2} \cos \varkappa_m \frac{y}{L},$$

where, as usual,

$$A_m = \frac{N_m}{\| V_0 \|^2}.$$

On the other hand, as we can show without difficulty, the solution of problem (4-46) in closed form

$$w = \frac{L^2}{\beta^2} - \frac{L^2}{\beta^2} \frac{\operatorname{ch} \beta \frac{y}{L}}{\beta \operatorname{sh} \beta + \mathrm{Bi} \operatorname{ch} \beta},$$

where

$$\mathrm{Bi} = hL.$$

Due to the uniqueness of the solution

$$\sum_{m=1}^{\infty} \frac{A_m}{\varkappa_m^2 + \beta^2} \cos \varkappa_m \frac{y}{L} = \frac{1}{\beta^2} - \frac{1}{\beta^2} \frac{\operatorname{ch} \beta \frac{y}{L}}{\beta \operatorname{sh} \beta + \mathrm{Bi} \operatorname{ch} \beta}. \tag{4-47}$$

Consequently

$$S_1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi\frac{x}{R}}{n} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi\frac{x}{R}}{n}\times$$

$$\times\frac{\operatorname{ch} n\pi\frac{y}{R}}{n\pi\frac{L}{R}\operatorname{sh} n\pi\frac{L}{R} + \mathrm{Bi}\operatorname{ch} n\pi\frac{L}{R}}.$$

However, we know that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nz}{n} = \frac{z}{2}\quad(-\pi < z < \pi).$$

Therefore

$$S_1 = \frac{x}{R} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi\frac{x}{R}}{n}\,\frac{\operatorname{ch} n\pi\frac{y}{R}}{n\pi\frac{L}{R}\operatorname{sh} n\pi\frac{L}{R} + \mathrm{Bi}\operatorname{ch} n\pi\frac{L}{R}}. \qquad (4\text{-}43'')$$

In contrast to series (4-43'), series (4-43") converges quite well where L > R in the entire area with the exception of points lying near the boundary y = L. Depending on the purposes of the investigation, we can use various representations of series S_1, which is important in calculation of the temperature.

2. Series S_2.

Series S_2 can be written as

$$S_2 = \frac{2}{\pi}\sum_{m=1}^{\infty}A_m\cos\varkappa_m\frac{y}{L}\sum_{n=1}^{\infty}\frac{n\sin nz}{n^2+p^2},$$

where

$$z = \frac{\pi x}{R};\quad p^2 = \frac{\varkappa_m^2 R^2}{\pi^2 L^2}.$$

However

$$\sum_{n=1}^{\infty} \frac{n \sin nz}{n^2+p^2} = \frac{\pi}{2} \frac{\operatorname{sh} p(\pi - z)}{\operatorname{sh} p\pi} \quad (0 < z < 2\pi). \tag{4-48}$$

Therefore

$$S_2 = \sum_{m=1}^{\infty} A_m \frac{\operatorname{sh} \varkappa_m \frac{R}{L}\left(1 - \frac{x}{R}\right)}{\operatorname{sh} \varkappa_m \frac{R}{L}} \cos \varkappa_m \frac{y}{L}. \tag{4-49}$$

Series (4-49) converges well were R > L everywhere with the exception of points near the boundary x = R (and also at the boundary x = R, where the series, as follows from formula (4-48), is not defined).

Another representation of series S_2 is produced if we write it as

$$S_2 = \frac{2\pi}{R^2} \sum_{n=1}^{\infty} n \sin n\pi \frac{x}{R} \sum \frac{A_m}{\left(\frac{n^2\pi^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right)} \cos \varkappa_m \frac{y}{L}.$$

Considering the summation formula (4-47), we have:

$$S_2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{x}{R}}{n} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{x}{R}}{n} \frac{\operatorname{ch} n\pi \frac{y}{R}}{n\pi \frac{L}{R} \operatorname{sh} n\pi \frac{L}{R} + \mathrm{Bi} \operatorname{ch} n\pi \frac{L}{R}}.$$

However

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n} = \frac{\pi - z}{2} \quad (0 < z < 2\pi).$$

Consequently

$$S_2 = 1 - \frac{x}{R} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{x}{R}}{n} \frac{\operatorname{ch} n\pi \frac{y}{R}}{n\pi \frac{L}{R} \operatorname{sh} n\pi \frac{L}{R} + \mathrm{Bi} \operatorname{ch} n\pi \frac{L}{R}}. \tag{4-50}$$

All which we have said earlier concerning the convergence of series (4-43") applies equally to series (4-50).

3. The rectangle $(0 < x < R,\ 0 < y < L)$. The initial temperature is a function of the coordinates. Heterogeneous boundary conditions of the third kind are assigned at the faces, the ambient temperature depends on time. There is heat liberation in the body, the intensity of which is described by the generalized function $q(\tau, T)$:

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{c\gamma}\ q(\tau, T) \quad (0 < x < R,\ 0 < y < L,\ \tau > 0);$$

$$T(x, y, 0) = f(x, y) \quad (0 \leqslant x \leqslant R,\ 0 \leqslant y \leqslant L);$$

$$\frac{\partial T((j-1)R, y, \tau)}{\partial x} = (-1)^j h_j [\psi_j(\tau) - T((j-1)R, y, \tau)] \quad (j = 1, 2);$$

$$\frac{\partial T(x, (i-3)L, \tau)}{\partial y} = (-1)^i h_i [\psi_i(\tau) - T(x, (i-3)L, \tau)] \quad (i = 3, 4). \tag{4-51}$$

Here $q(\tau, T) = q_\nu(d_\nu + b_\nu T)e^{-m_\nu \tau}$, where q_ν, d_ν, b_ν, m_ν are parameters, piecewise-constant functions of time, defined in $(\tau_{\nu-1}, \tau_\nu)$ $(\nu = 1, 2, \ldots, s)$.

The use of repeated finite integral transforms using the substitution functions recommended in § 3-3 with F functions of the first kind allows us to produce a solution to problem (4-51) in a form similar to algorithm (4-9):

$$T = \Phi(y, \tau) - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\|U_0\|^2 \|V_0\|^2}\, U_0\left(\mu_n \frac{x}{R}\right) V_0\left(\varkappa_m \frac{y}{L}\right) \times$$
$$\times \exp\left[-\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right) a\tau_s\right] \exp\left[\sum_{\nu=1}^{s} c_\nu\right] \left\{B_{nm} - \right.$$
$$\left. - \sum_{\nu=1}^{s} \Psi_{nm\nu} \exp\left[-\sum_{\nu'=1}^{\nu} c_{\nu'}\right]\right\}, \tag{4-52}$$

where

$$\Phi(y, \tau) = \psi_3 + \frac{(\psi_3 - \psi_4)\,\mathrm{Bi}_3}{(\mathrm{Bi}_3 + \mathrm{Bi}_4 + \mathrm{Bi}_3 \mathrm{Bi}_4)} \left(1 + \mathrm{Bi}_4 - \mathrm{Bi}_4 \frac{y}{L}\right);$$

$$B_{nm} = \psi_1(0) N_n N_m + [\psi_3(0) - \psi_2(0)]\, \frac{\mathrm{Bi}_3 L}{\varkappa_m} N_n -$$
$$- \int_0^R \int_0^L f(x, y)\, U_0\left(\mu_n \frac{x}{R}\right) V_0\left(\varkappa_m \frac{y}{L}\right) dx\, dy;$$

$$\Psi_{nm\nu} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu}\right] \int\limits_{\tau_{\nu-1}}^{\tau_\nu} L_\nu(\zeta) \exp\left[\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right) a\zeta - \right.$$

$$\left. - m_\nu \zeta + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta}\right] d\zeta;$$

$$L_\nu(\tau) = \frac{a\mu_n^2}{R^2} \left[(\psi_2 - \psi_4) N_n N_m + (\psi_1 - \psi_2) \frac{\mathrm{Bi}_1 R}{\mu_n} N_m - \right.$$

$$\left. - (\psi_3 - \psi_4) \frac{\mathrm{Bi}_3 L}{\varkappa_m} N_n\right] e^{m_\nu \tau} + \frac{q_\nu h_\nu}{c\gamma}\left[(\psi_3 - \psi_4) \frac{\mathrm{Bi}_3 L}{\varkappa_m} N_n + \psi_4 N_n N_m\right] +$$

$$+ \frac{q_\nu d_\nu}{c\gamma} N_n N_m - \left[(\psi'_3 - \psi'_4) \frac{\mathrm{Bi}_3 L}{\varkappa_m} N_n + \psi'_4 N_n N_m\right] e^{m_\nu \tau};$$

$$\| U_0 \|^2 = \frac{R}{2}\left[\mathrm{Bi}_1^2 + \mathrm{Bi}_1 + \mu_n^2 + \frac{\mathrm{Bi}_2 (\mathrm{Bi}_1^2 + \mu_n^2)}{\mathrm{Bi}_2^2 + \mu_n^2}\right];$$

$$N_n = \frac{R}{\mu_n}\left[\mathrm{Bi}_2 + (-1)^{n+1} \mathrm{Bi}_2 \sqrt{\frac{\mathrm{Bi}_1^2 + \mu_n^2}{\mathrm{Bi}_2^2 + \mu_n^2}}\right];$$

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R};$$

μ_n is the root of the characteristic equation

$$\cot \mu_n = \frac{\mu_n^2 - \mathrm{Bi}_1 \mathrm{Bi}_2}{\mu_n (\mathrm{Bi}_1 + \mathrm{Bi}_2)}; \quad \mathrm{Bi}_1 = h_1 R; \quad \mathrm{Bi}_2 = h_2 R.$$

The functions $\|V_0\|^2$, N_m, $V_0(\kappa_m \frac{y}{L})$ and the root κ_m have the same form as $\|U_0\|^2$, N_n, $U_0(\mu_n \frac{x}{R})$ and μ_n, but relate to coordinate y.

If the ambient temperature at all faces of the rectangle is identical and depends only on time $\psi = \psi(\tau)$, the relative heat exchange factors on the opposite faces are identical ($h_1 = h_2 = h_R$, $h_3 = h_4 = h_L$) and the initial temperature f(x, y) is a symmetrical function, the solution of the problem, produced in a system of coordinates with its origin at the center of the rectangle becomes

$$T = \psi(\tau) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \cos \mu_n \frac{x}{R} \cos \varkappa_m \frac{y}{L} \exp\left[-\left(\frac{\mu_n^2}{R^2} + \right.\right.$$

$$\left.\left. + \frac{\varkappa_m^2}{L^2}\right) a\tau_s\right] \exp\left[\sum_{\nu=1}^{s} c_\nu\right] \left\{ B_{nm} - \sum_{\nu=1}^{s} \Psi_{nm\nu} \exp\left[-\sum_{\nu'=1}^{\nu} c_{\nu'}\right]\right\}, \qquad (4\text{-}53)$$

where

$$B_{nm} = \psi(0) - \frac{1}{N_n N_m} \int_0^R \int_0^L f(x, y) \cos \mu_n \frac{x}{R} \cos \varkappa_m \frac{y}{L} \, dx\, dy;$$

$$\Psi_{nm\nu} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu} \right] \int_{\tau_{\nu-1}}^{\tau_\nu} L(\zeta) \exp\left[\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2} \right) a\zeta - \right.$$

$$\left. - m_\nu \zeta + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta} \right] d\zeta;$$

$$\mathcal{L}_\nu(\zeta) = \frac{q_\nu d_\nu}{c\gamma} + \frac{q_\nu b_\nu}{c\gamma} \psi(\zeta) - \psi'(\zeta) e^{m_\nu \zeta};$$

$$N_n = \frac{(-1)^{n+1} \mathrm{Bi}_R}{\mu_n (\mathrm{Bi}_R^2 + \mu_n^2)}, \quad N_m = \frac{(-1)^{m+1} \mathrm{Bi}_L}{\varkappa_m (\mathrm{Bi}_L^2 + \varkappa_m^2)}; \; \mathrm{Bi}_R = h_R R; \; \mathrm{Bi}_L = h_L L;$$

$$A_n = \frac{(-1)^{n+1} 2\mathrm{Bi}_R \sqrt{\mathrm{Bi}_R^2 + \mu_n^2}}{\mu_n (\mathrm{Bi}_R^2 + \mathrm{Bi}_R + \mu_n^2)}; \quad A_m = \frac{(-1)^{m+1} 2\mathrm{Bi}_L \sqrt{\mathrm{Bi}_L^2 + \varkappa_m^2}}{\varkappa_m (\mathrm{Bi}_L^2 + \mathrm{Bi}_L + \varkappa_m^2)};$$

μ_n and κ_m are the roots of the characteristic equation

$$\cot \mu_n = \frac{\mu_n}{\mathrm{Bi}_R}; \quad \cot \varkappa_m = \frac{\varkappa_m}{\mathrm{Bi}_L}.$$

4. The rectangle (-R < x < R, -L < y < L). The initial and boundary conditions are homogeneous (for definition, let us analyze boundary conditions of the first kind). There is heat liberation in the body, depending exponentially on time:

$$\frac{\partial T}{\partial \tau} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{q_0}{c\gamma} e^{-\beta\tau} \quad (0 < x < R, \; 0 < y < L, \; \tau > 0);$$

$$T(x, y, 0) = 0 \quad (0 \leqslant x \leqslant R, \; 0 \leqslant y \leqslant L);$$

$$T(R, y, \tau) = T(x, L, \tau) = 0;$$

$$\frac{\partial T(0, y, \tau)}{\partial x} = \frac{\partial T(x, 0, \tau)}{\partial y} = 0 \quad \text{(condition of symmetry)}.$$

The use of double finite integral transforms gives us

$$\frac{d\bar{\bar{T}}_{nm}}{d\tau} = -a \left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2} \right) \bar{\bar{T}}_{nm} + \frac{q_0}{c\gamma} e^{-\beta\tau} N_n N_m;$$

$$\bar{\bar{T}}_{nm}(0) = 0.$$

From this it follows that

$$\bar{\bar{T}}_{nm} = \frac{q_0}{c\gamma}\frac{N_n N_m}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]}e^{-\beta\tau}-\frac{q_0}{c\gamma}\frac{N_n N_m}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]}\times$$

$$\times e^{-\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a\tau}$$

and on the basis of the corresponding inversion formulas

$$T(x, y, \tau) = \frac{q_0}{c\gamma}e^{-\beta\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_n A_m U_0\left(\mu_n\frac{x}{R}\right)V_0\left(\varkappa_m\frac{y}{L}\right)}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]}-$$

$$-\frac{q_0}{c\gamma}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_n A_m U_0\left(\mu_n\frac{x}{R}\right)V_0\left(\varkappa_m\frac{y}{L}\right)}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]}e^{-\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a\tau}. \qquad (4\text{-}54)$$

The symbols used here are obvious from the preceding text.

In formula (4-54), we transform the first double sum

$$S_1 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_n A_m U_0\left(\mu_n\frac{x}{R}\right)V_0\left(\varkappa_m\frac{y}{L}\right)}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]}.$$

Let us study:

$$S_2 = \sum_{m=1}^{\infty}\frac{A_m V_0\left(\varkappa_m\frac{y}{L}\right)}{\left[\left(\frac{\mu_n^2}{R^2}+\frac{\varkappa_m^2}{L^2}\right)a-\beta\right]} = \frac{L^2}{a}\sum_{m=1}^{\infty}\frac{N_m V_0\left(\varkappa_m\frac{y}{L}\right)}{\|V_0\|^2(\varkappa_m^2+m^{*2})},$$

where

$$m^{*2} = \left(\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L\right); \quad \beta^{*2}_L = \frac{\beta L^2}{a}.$$

In accordance with the data of § 3-3, the sum of the series

$$L^2 \sum_{m=1}^{\infty} \frac{N_m V_0 \left(\varkappa_m \frac{y}{L}\right)}{\| V_0 \|^2 (\varkappa_m^2 + m^{*2})}$$

is the integral of the differential equation

$$\frac{d^2 w}{dy^2} - \frac{m^2}{R^2} w = -1$$

with the boundary conditions

$$\frac{dw(0)}{dy} = 0; \quad w(L) = 0.$$

Consequently

$$S_2 = \frac{L^2}{a\left(\mu_n \frac{L^2}{R^2} - \beta^{*2}_L\right)} \left(1 - \frac{\operatorname{ch} \sqrt{\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L}\, \frac{y}{L}}{\operatorname{ch} \sqrt{\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L}}\right).$$

We can similarly show that

$$S_3 = \sum_{n=1}^{\infty} \frac{A_n U_0 \left(\mu_n \frac{x}{R}\right)}{\left[\left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2}\right) a - \beta\right]} = \frac{R^2}{a\left(\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R\right)} \times$$

$$\times \left(1 - \frac{\operatorname{ch} \sqrt{\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R}\, \frac{y}{L}}{\operatorname{ch} \sqrt{\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R}}\right); \quad \beta^{*2}_R = \frac{\beta R^2}{a}.$$

Thus

$$S_1 = \frac{L^2}{a} \sum_{n=1}^{\infty} \frac{A_n U_0\left(\mu_n \frac{x}{R}\right)}{\left(\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L\right)} \left(1 - \frac{\operatorname{ch} \sqrt{\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L}\,\frac{y}{L}}{\operatorname{ch} \sqrt{\mu_n^2 \frac{L^2}{R^2} - \beta^{*2}_L}}\right) =$$

$$= \frac{R^2}{a} \sum_{m=1}^{\infty} \frac{A_m V_0\left(\varkappa_m \frac{y}{L}\right)}{\left(\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R\right)} \left(1 - \frac{\operatorname{ch} \sqrt{\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R}\,\frac{x}{R}}{\operatorname{ch} \sqrt{\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R}}\right).$$

However, the sum

$$S_4 = L^2 \sum_{n=1}^{\infty} \frac{A_n U_0\left(\mu_n \frac{x}{R}\right)}{\left(\mu_n \frac{L^2}{R^2} - \beta^{*2}_L\right)} = R^2 \sum_{n=1}^{\infty} \frac{A_n U_0\left(\mu_n \frac{x}{R}\right)}{(\mu_n^2 - \beta^{*2}_R)}$$

on the strength of what we have said earlier, is an integral of the differential equation

$$\frac{d^2 v}{dx^2} + \frac{\beta^{*2}_R}{R^2} v = -1$$

with homogeneous boundary conditions, i.e.

$$S_4 = \frac{R^2}{\beta^{*2}_R} \left(\frac{\cos \beta^*_R \frac{x}{R}}{\cos \beta^*_R} - 1\right).$$

Similarly,

$$S_5 = R^2 \sum_{m=1}^{\infty} \frac{A_m V_0\left(\varkappa_m \frac{y}{L}\right)}{\left(\varkappa_m^2 \frac{R^2}{L^2} - \beta^{*2}_R\right)} = \frac{L^2}{\beta^{*2}_L} \left(\frac{\cos \beta^*_L \frac{y}{L}}{\cos \beta^*_L} - 1\right).$$

Then finally

$$S_1 = \frac{R^2}{a\beta^{*2}_R}\left(\frac{\cos\beta^*_R\frac{x}{R}}{\cos\beta^*_R} - 1\right) -$$

$$-\frac{L^2}{a}\sum_{n=1}^{\infty}\frac{A_n U_0\left(\mu_n\frac{x}{R}\right)\operatorname{ch}\sqrt{\mu_n^2\frac{L^2}{R^2}-\beta^{*2}_L}\frac{y}{L}}{\left(\mu_n^2\frac{L^2}{R^2}-\beta^{*2}_L\right)\operatorname{ch}\sqrt{\mu_n^2\frac{L^2}{R^2}-\beta^{*2}_L}} =$$

$$= \frac{L^2}{a\beta^{*2}_L}\left(\frac{\cos\beta^*_L\frac{y}{L}}{\cos\beta^*_L} - 1\right) -$$

$$-\frac{R^2}{a}\sum_{m=1}^{\infty}\frac{A_m V_0\left(\varkappa_m\frac{y}{L}\right)\operatorname{ch}\sqrt{\varkappa_m^2\frac{R^2}{L^2}-\beta^{*2}_R}\frac{x}{R}}{\left(\varkappa_m^2\frac{R^2}{L^2}-\beta^{*2}_R\right)\operatorname{ch}\sqrt{\varkappa_m^2\frac{R^2}{L^2}-\beta^{*2}_R}}.$$

A finite solid cylinder ($0 < r < R$, $0 < z < H$).

The initial temperature T_0. The end surfaces are maintained at temperatures T_1 and T_2; on the cylindrical surface heat exchange with the environment is by convection. The ambient temperature is T_3. In the lower portion of the cylinder ($0 < r < R$, $0 < z < L$), heat is liberated, the intensity of heat liberation is constant

$$\frac{\partial T}{\partial\tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \frac{1}{c\gamma}q(z) \quad (0<r<R,\ 0<z<H,\ \tau>0); \tag{4-55}$$

$$\begin{gathered} T(r,\ z,\ 0) = T_0 \quad (0\leqslant r\leqslant R,\ 0\leqslant z\leqslant H); \\ T(r,\ 0,\ \tau) = T_1;\ T(r,\ H,\ \tau) = T_2; \\ \frac{\partial T(R,\ z,\ \tau)}{\partial r} = h[T_3 - T(R,\ z,\ \tau)];\ T(0,\ z,\ \tau) \neq \infty. \end{gathered} \tag{4-56}$$

Here

$$q(z) = \begin{cases} q_0 & \text{where } 0\leqslant z\leqslant L; \\ 0 & \text{where } z > L. \end{cases}$$

Since heat liberation in the cylinder is a function of z, in solving the problem it is convenient first to use a finite integral transform with respect to this coordinate.

The finite integral transform with respect to z in interval (0, H) with boundary conditions of the first kind is determined by the formulas:

$$\tilde{u}_n = \int_0^H u(z)\, U_0\left(\mu_n \frac{z}{H}\right) dz;$$

$$u(z) = \sum_{n=1}^{\infty} \frac{\tilde{u}_n}{\|U_0\|^2} U_0\left(\mu_n \frac{z}{H}\right), \tag{4-57}$$

where μ_n is the root of the characteristic equation

$$\sin \mu_n = 0; \quad \mu_n = n\pi \, (n = 1, \; 2, \; \ldots);$$

$U_0(\mu_n \frac{z}{H}) = \sin \mu_n \frac{z}{H}$ is the Eigenfunction of the problem; $\|U_0\|^2 = H/2$.

Boundary conditions (4-56) are heterogeneous. We therefore assume

$$\vartheta = \Phi(z) - T,$$

where

$$\Phi(z) = T_1 + (T_2 - T_1) F_{1a}; \; F_{1a} = \frac{z}{H}.$$

From this we have:

$$\frac{\partial \vartheta}{\partial \tau} = a \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \vartheta}{\partial r} \right) + \frac{\partial^2 \vartheta}{\partial z^2} \right] - \frac{1}{c\gamma} q(z);$$

$$\vartheta(r, \; z, \; 0) = M(z);$$

$$\vartheta(r, \; 0, \; \tau) = \vartheta(r, \; H, \; \tau) = 0;$$

$$\frac{\partial \vartheta(R, \; z, \; \tau)}{\partial r} = h\,[\Psi(z) - \vartheta(R, \; z, \; \tau)]; \; \vartheta(0, \; z, \; \tau) \neq \infty.$$

Here

$$M(z) = (T_1 - T_0) + (T_2 - T_1) F_{1a};$$
$$\Psi(z) = (T_1 - T_3) + (T_2 - T_1) F_{1a}.$$

After performing integral transform (4-57), we produce:

$$\frac{\partial \tilde{\vartheta}_n}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{\vartheta}_n}{\partial r}\right)\right] - \frac{a\mu_n^2}{H^2}\tilde{\vartheta}_n - K_n;$$
$$\tilde{\vartheta}_n(r, 0) = \tilde{M}_n;$$
$$\frac{\partial \tilde{\vartheta}_n(R, \tau)}{\partial r} = h[\tilde{\Psi}_n - \tilde{\vartheta}_n(R, \tau)];\ \tilde{\vartheta}_n(0, \tau) \neq \infty,$$

where

$$K_n = \frac{q_0}{c\gamma}\int_0^L \sin\mu_n\frac{z}{H}\,dz = -\frac{q_0 H}{\mu_n c\gamma}\left(1 - \cos\mu_n\frac{L}{H}\right);$$
$$\tilde{M}_n = (T_1 - T_0) N_n + (T_2 - T_1)\tilde{F}_{1a};$$
$$\tilde{\Psi}_n = (T_1 - T_3) N_n + (T_2 - T_1)\tilde{F}_{1a};$$
$$N_n = \int_0^H \sin\mu_n\frac{z}{H}\,dz = \frac{2H}{(2n-1)\pi};$$

$\tilde{F}_{1a} = (-1)^{n+1}\frac{H}{\mu_n}$ (see reference data of § 4-2).

We can now apply to the problem for $\tilde{\theta}_n$ a Laplace transform

$$\bar{\tilde{\vartheta}}_n(r, s) = \int_0^\infty \tilde{\vartheta}_n(r, \tau)\, e^{-s\tau}\, d\tau.$$

This gives us the supplemental equation

$$\frac{d^2\bar{\tilde{\vartheta}}_n}{dr^2} + \frac{1}{r}\frac{d\bar{\tilde{\vartheta}}_n}{dr} - \left(\frac{s}{a} + \frac{\mu_n^2}{H^2}\right)\bar{\tilde{\vartheta}}_n = \frac{1}{as}K_n - \frac{1}{a}\tilde{M}; \qquad (4\text{-}58')$$

the boundary conditions

$$\frac{d\bar{\bar{\theta}}_n(R, s)}{dr} = h\left[\frac{1}{s}\bar{\Psi}_n - \bar{\bar{\theta}}_n(R, s)\right]; \quad \bar{\bar{\theta}}_n(0, s) \neq \infty. \qquad (4\text{-}58)$$

The solution of the heterogeneous modified Bessel equation (4-58') [19, 160] is:

$$\bar{\bar{\theta}}_n = AI_0(pr) + BK_0(pr) + \frac{\bar{M}_n}{s + \alpha_n^2} - \frac{K_n}{s(s + \alpha_n^2)},$$

where

$$\alpha_n = \frac{a\mu_n^2}{H^2}; \quad p = \sqrt{\frac{s}{a} + \frac{\mu_n^2}{H^2}},$$

$I_\nu(pr)$, $K_\nu(pr)$ are modified Bessel functions of the first and second kind, of νth order.

Satisfying boundary conditions (4-58)[1], we find:

$$\bar{\bar{\theta}}_n = \bar{u}_1(s) + \bar{\Psi}_n\bar{u}_2(s) - \bar{M}_n\bar{u}_3(s) + K_n\bar{u}_4(s),$$

where

$$\bar{u}_1(s) = \frac{\bar{M}_n}{s + \alpha_n^2} - \frac{K_n}{s(s + \alpha_n^2)};$$

$$\bar{u}_2(s) = \frac{1}{s\Delta(s)} I_0(pr);$$

$$\bar{u}_3(s) = \frac{1}{(s + \alpha_n^2)\Delta(s)} I_0(pr);$$

$$\bar{u}_4(s) = \frac{1}{s(s + \alpha_n^2)\Delta(s)} I_0(pr);$$

$$\Delta(s) = I_0(pr) + \frac{1}{h} pI_1(pr).$$

---

[1]Considering the finite nature of function $\bar{\bar{\theta}}_n$ on the axis of the cylinder r = 0.

From this

$$\vartheta_n = \mathcal{L}^{-1}[\bar{\vartheta}_n] = \mathcal{L}^{-1}[\bar{u}_1(s)] + \tilde{\Psi}_n \mathcal{L}^{-1}[\bar{u}_2(s)] -$$
$$- \tilde{M}_n \mathcal{L}^{-1}[\bar{u}_3(s)] + K_n \mathcal{L}^{-1}[\bar{u}_4(s)].$$

1. The mapping $\bar{u}_1(s)$ satisfies the conditions of the first theorem of the expansion (see § 3-4) and therefore

$$\mathcal{L}^{-1}[\bar{u}_1(s)] = u_1(\tau) = \tilde{M}_n e^{-\alpha_n^2 \tau} - \frac{K_n}{\alpha_n^2}(1 - e^{-\alpha_n^2 \tau})$$

or

$$u_1(\tau) = \tilde{M}_n e^{-\mu_n^2 \frac{a\tau}{H^2}} - \frac{K_n H^2}{a\mu_n^2}\left(1 - e^{-\mu_n^2 \frac{a\tau}{R^2}}\right).$$

2. According to the Riemann-Mellin inversion formula

$$\mathcal{L}^{-1}[\bar{u}_2(s)] = u_2(\tau) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{I_0(pr)}{s\left[I_0(pR) + \frac{1}{h} p I_1(pr)\right]} e^{s\tau} ds,$$

where

$$p = \sqrt{\frac{s}{a} + \frac{\mu_n^2}{H^2}}.$$

The integrand $\bar{u}_2(s)e^{s\tau}$ is an unambiguous function of the complex variable s. This can be seen, for example, from the expansion of modified Bessel functions $I_0(z)$ and $I_1(z)$ into series [63, 69, 113].

Function $\bar{u}_2(s)$ is meromorphic, its singular points are the simple pole s = 0 and the roots s_m of the equation

$$\Delta(s_m) = I_0(p_m R) + \frac{1}{h} p_m I_1(p_m R) = 0.$$

We assume:

$$ip_m R = iR\sqrt{\frac{s_m}{a} + \frac{\mu_n^2}{H^2}} = \varkappa_m,$$

so that

$$s_m = -\frac{a\mu_n^2}{H^2} - \frac{a\varkappa_m^2}{R^2},$$

and keep in mind the relationships

$$\mathrm{I}_0(z) = J_0(iz);\ \mathrm{I}_1(z) = -iJ_1(iz).$$

Then equation $\Delta(s_m) = 0$ can be rewritten as

$$\Delta(s_m) = \mathrm{I}_0(p_m R) + \frac{1}{h}p_m \mathrm{I}_1(p_m R) = J_0(\varkappa_m) - \frac{\varkappa_m}{\mathrm{Bi}}J_1(\varkappa_m) = 0$$

or

$$\frac{J_0(\varkappa_m)}{J_1(\varkappa_m)} = \frac{\varkappa_m}{\mathrm{Bi}};\ \mathrm{Bi} = hR. \tag{4-59}$$

Expression (4-59) is the characteristic equation for the problem of heat conductivity for a solid cylinder with boundary conditions of the third kind. Its roots κ_m form an infinite increasing sequence of real positive numbers.

The integration contour is taken in a form similar to that shown, for example, in Figure 3-2. The radii of the circular arcs C_{R_m} are assumed equal to

$$B_m = \frac{a\eta_m^2}{R^2},\ \text{где}\ \eta_m^2 = \frac{\varkappa_m^2 + \varkappa_{m+1}^2}{2}\ (m = 1,\ 2 \ldots).$$

As we did earlier, we can assume that the function

$$\Phi(s) = \frac{1}{\Delta(s)} I_0(pr)$$

in arcs C_{B_m} is limited.

Consequently, here function $\bar{u}_2(s)$ has order $O(R^{-1})$ and therefore satisfies the conditions of the Jordan lemma; furthermore, it is absolutely integrable on line $(\sigma - i\rho_m, \sigma + i\rho_m)$.

In the case which we are studying, the conditions of the second theorem of the expansion are also fulfilled, because

$$u_2(\tau) = \sum_{s_j} \operatorname{res} \bar{u}_2(s_j) e^{s_j \tau}$$

In the vicinity of both singular points

$$\bar{u}_2 = \frac{\varphi(s)}{\psi(s)},$$

where $\phi(s)$ and $\psi(s)$ are analytic functions, equal to

$$\varphi(s) = I_0(pr);$$
$$\psi(s) = s\Delta(s) = s\left[I_0(pR) + \frac{1}{h} pI_1(pR)\right].$$

From this

$$u_2(\tau) = \sum_{s_j} \frac{\varphi(s_j)}{\psi'(s_j)} e^{s_j \tau}.$$

Derivative $\psi'(s)$ is equal to:

$$\psi'(s) = \Delta(s) + s[\Delta(s)]' = \left[I_0(pR) + \frac{1}{h} pI_1(pR)\right] +$$
$$+ \frac{R}{2a} s\left[\frac{1}{p} I_1(pR) + \frac{1}{h} I_0(pR)\right].$$

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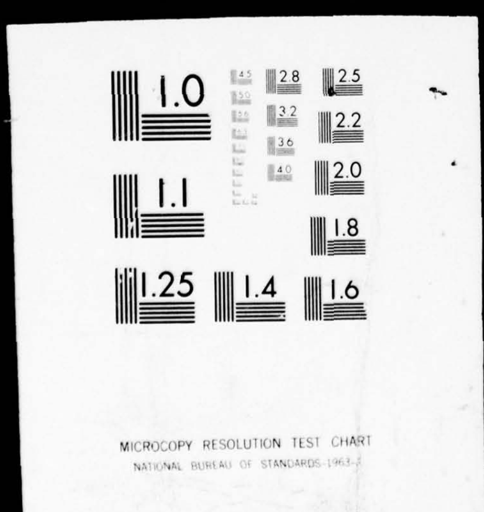

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Then

$$\underset{s\to 0}{\varphi(s)} = 1; \; \underset{s\to 0}{\psi(s)} = I_0\left(\mu_n \frac{R}{H}\right) + \frac{1}{Bi}\frac{R}{H} I_1\left(\mu_n \frac{R}{H}\right);$$

$$\underset{s\to s_m}{\varphi(s)} = I_0(p_m r) = J_0\left(\varkappa_m \frac{r}{R}\right);$$

$$\underset{s\to s_m}{\psi'(s)} = -\frac{1}{2}\left(\frac{\mu_n^2}{H^2} + \frac{\varkappa_m^2}{R^2}\right)\frac{J_0(\varkappa_m)}{Bi\,\dfrac{\varkappa_m^2}{R^2}}(\varkappa_m^2 + Bi^2).$$

In producing the relationship for $\psi'(s)$ $(s \to s_m)$, we considered the characteristic equation (4-59).

Thus

$$u_3(\tau) = \frac{1}{\left[I_0\left(\mu_n \frac{R}{H}\right) + \frac{1}{Bi}\mu_n \frac{R}{H} I_1\left(\mu_n \frac{R}{H}\right)\right]} - \sum_{m=1}^{\infty} \frac{A_m \dfrac{\varkappa_m^2}{R^2}}{\left(\dfrac{\mu_n^2}{H^2} + \dfrac{\varkappa_m^2}{R^2}\right)} \times$$

$$\times J_0\left(\varkappa_m \frac{r}{R}\right) \exp\left[-\left(\frac{\mu_n^2}{H^2} + \frac{\varkappa_m^2}{R^2}\right) a\tau\right],$$

where

$$A_m = \frac{2Bi}{J_0(\varkappa_m)\,[Bi^2 + \varkappa_m^2]}.$$

3. The mapping $u_3(s)$ is equal to:

$$\bar{u}_3(s) = \frac{1}{(s + a_n^2)\,\Delta(s)} I_0(pr)$$

or, in accordance with the symbols now being used

$$\bar{u}_3(s) = \frac{1}{ap^2 \Delta(s)} I_0(pr).$$

It follows from the result produced for function $\bar{u}_2(s)$ that the mapping

$$\bar{v}(s) = \frac{I_0\left(\sqrt{\frac{s}{a}}\,r\right)}{s\left[I_0\sqrt{\frac{s}{a}}\,R + \frac{1}{h}\sqrt{\frac{s}{a}}\,I_1\left(\sqrt{\frac{s}{a}}\,R\right)\right]}$$

has the original

$$\mathscr{L}^{-1}[\bar{v}(s)] = v(\tau) = 1 - \sum_{m=1}^{\infty} A_m J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\varkappa_m^2 \frac{a\tau}{R^2}}.$$

From this, using the properties of similarity and displacement (see § 3-4), we find

$$\mathscr{L}^{-1}[\bar{u}_3(s)] = u_3(\tau) = e^{-\mu_n^2 \frac{a\tau}{H^2}} - \sum_{m=1}^{\infty} A_m J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau},$$

where

$$\delta_{nm}^2 = \frac{\mu_n^2}{H^2} + \frac{\varkappa_m^2}{R^2}.$$

4. Since

$$\bar{u}_4(s) = \frac{I_0(pr)}{s(s+\alpha_n^2)\Delta(s)} = \frac{I_0(pr)}{\alpha_n^2 s\Delta(s)} - \frac{I_0(pr)}{\alpha_n^2(s+\alpha_n^2)\Delta(s)},$$

then, obviously,

$$\mathscr{L}^{-1}[\bar{u}_4(s)] = u_4(\tau) = \frac{H^2}{a\mu_n^2 W\left(\mu_n \frac{R}{H}\right)} - \frac{H^2}{aR^2}\sum_{m=1}^{\infty}\frac{\varkappa_m^2 A_m}{\mu_n^2 \delta_{nm}^2}\times$$

$$\times J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau} - \frac{H^2}{a\mu_n^2} e^{-\mu_n^2 \frac{a\tau}{H^2}} + \frac{H^2}{a}\sum_{m=1}^{\infty}\frac{A_m}{\mu_n^2} J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau},$$

where

$$W\left(\mu_n \frac{R}{H}\right) = I_0\left(\mu_n \frac{R}{H}\right) + \frac{\mu_n}{\mathrm{Bi}} I_1\left(\mu_n \frac{R}{H}\right).$$

Consequently, transform $\tilde{\theta}_n$ is equal to

$$\tilde{\theta}_n = \frac{\tilde{\Psi}_n}{W\left(\mu_n \frac{R}{H}\right)} + \sum_{m=1}^{\infty} A_m J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm} a\tau}\left[\tilde{M}_n - \frac{\tilde{\Psi}_n \varkappa_m^2}{R^2 \delta_{nm}^2}\right] - \frac{K_n H^2}{a\mu_n^2} + \frac{K_n H^2}{a\mu_n^2 W\left(\mu_n \frac{R}{H}\right)} - \frac{H^2}{a}\sum_{m=1}^{\infty} \frac{K_n A_m}{\mu_n^2}\left[\frac{\varkappa_m^2}{R^2\delta_{nm}^2} - 1\right] \times$$
$$\times J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau}.$$

Based on the inversion formulas from (4-57) considering the substitutions made and symbols introduced, we find:

$$T = T_1 + (T_2 - T_1)\frac{z}{H} - (T_2 - T_1)\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2 \sin \mu_n \frac{z}{H}}{\mu_n W\left(\mu_n \frac{R}{H}\right)} -$$
$$- (T_1 - T_3)\sum_{n=1}^{\infty} \frac{4 \sin \mu_n \frac{z}{H}}{(2n-1)\pi W\left(\mu_n \frac{R}{H}\right)} + \frac{q_0 H^2}{\lambda}\sum_{n=1}^{\infty} \frac{2}{\mu_n^3}\left(1 - \cos \mu_n \frac{L}{H}\right) \times$$
$$\times \sin \mu_n \frac{z}{H} - \frac{q_0 H^2}{\lambda}\sum_{n=1}^{\infty} \frac{2\left(1 - \cos \mu_n \frac{L}{H}\right)}{\mu_n^3 W\left(\mu_n \frac{R}{H}\right)} \sin \mu_n \frac{z}{H} +$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4A_m}{(2n-1)\pi}\left[(T_1 - T_0) - (T_1 - T_3)\frac{\varkappa_m^2}{R^2\delta_{nm}^2}\right] \sin \mu_n \frac{z}{H} J_0\left(\varkappa_m \frac{r}{R}\right) \times$$
$$\times e^{-\delta_{nm}^2 a\tau} - (T_2 - T_1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+1} 2A_m}{\mu_n}\left[\frac{\varkappa_m^2}{R^2\delta_{nm}^2} - 1\right] \sin \mu_n \frac{z}{H} \times$$
$$\times J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau} + \frac{q_0 H^2}{\lambda}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{2A_m\left(1 - \cos \mu_n \frac{L}{H}\right)}{\mu_n^3} \times$$
$$\times \left[\frac{\varkappa_m^2}{R^2\delta_{nm}^2} - 1\right] \sin \mu_n \frac{z}{H} J_0\left(\varkappa_m \frac{r}{R}\right) e^{-\delta_{nm}^2 a\tau}, \qquad (4\text{-}60)$$

where $\mu_n = n\pi$ $(n = 1, 2, \ldots)$; κ_m is the root of the characteristic equation

$$\frac{J_0(\kappa_m)}{J_1(\kappa_m)} = \frac{\kappa_m}{\mathrm{Bi}};\quad \mathrm{Bi} = hR;$$

$$A_m = \frac{2\mathrm{Bi}}{J_0(\kappa_m)\,[\mathrm{Bi}^2 + \kappa_m^2]}.$$

The series

$$\frac{q_0 H^2}{\lambda} \sum_{n=1}^{\infty} \frac{2\left(1 - \cos \mu_n \frac{L}{H}\right)}{\mu_n^3} \sin \mu_n \frac{z}{H},$$

included in solution (4-60) can be summed if we use the recommendations presented in § 3-3.

Let us study the solution of the following problem:

$$\frac{d^2 v}{dz^2} + \frac{1}{\lambda} q(z) = 0 \quad (0 < z < H); \tag{4-61}$$

$$v(0) = v(H) = 0, \tag{4-62}$$

where

$$q(z) = \begin{cases} q_0 & \text{where } 0 \leq z \leq L; \\ 0 & \text{where } z > L. \end{cases}$$

Applying the finite integral transform (4-57) to equation (4-61), considering the homogeneous boundary conditions (4-62), we find

$$\tilde{v}_n = \frac{q_0 H^2}{\lambda} \, \frac{H\left(1 - \cos \mu_n \frac{z}{H}\right)}{\mu_n^3}$$

and, on the basis of the inversion formula (4-57)

$$v = \frac{q_0 H^2}{\lambda} \sum_{n=1}^{\infty} \frac{H\left(1 - \cos \mu_n \frac{z}{H}\right)}{\mu_n^3 \, \| U_0 \|^2} U_0\left(\mu_n \frac{z}{H}\right)$$

or

$$v=\frac{q_0H^2}{\lambda}\sum_{n=1}^{\infty}\frac{2\left(1-\cos\mu_n\frac{z}{H}\right)}{\mu_n^3}\sin\mu_n\frac{z}{H}. \tag{4-63}$$

We now produce the solution of problem (4-61)-(4-62) in closed form. Let us formulate this problem as

$$\frac{d^2v_1}{dz^2}+\frac{q_0}{\lambda}=0\ (0<z<L);$$

$$\frac{d^2v_2}{dz^2}=0\ (L<z<H);$$

$$v_1(L)=v_2(L);\quad \frac{dv_1(L)}{dz}=\frac{dv_2(L)}{dz};$$

$$v_1(0)=v_2(H)=0.$$

From this

$$v_1=\frac{q_0H^2}{\lambda}\left[-\frac{1}{2}\left(\frac{z}{H}\right)^2-\frac{1}{2}\left(\frac{L}{H}\right)^2\frac{z}{H}+\frac{L}{H}\frac{z}{H}\right]\ (0<z<L);$$

$$v_2=\frac{q_0H^2}{\lambda}\left[-\frac{1}{2}\left(\frac{L}{H}\right)^2\frac{z}{H}+\left(\frac{L}{H}\right)^2\right]\ (L<z<H). \tag{4-64}$$

Comparing solutions (4-63) and (4-64), we see that

$$\frac{q_0H^2}{\lambda}\sum_{n=1}^{\infty}\frac{2\left(1-\cos\mu_n\frac{z}{H}\right)}{\mu_n^3}=$$

$$=\begin{cases}\frac{q_0H^2}{\lambda}\left[-\frac{1}{2}\left(\frac{z}{H}\right)^2-\frac{1}{2}\left(\frac{L}{H}\right)^2+\frac{L}{H}\frac{z}{H}\right]\ (0<z<L);\\ \frac{q_0H^2}{\lambda}\left[-\frac{1}{2}\left(\frac{L}{H}\right)^2\frac{z}{H}+\left(\frac{L}{H}\right)^2\right]\ (L<z<H).\end{cases}$$

4-4. Calculations of Temperature Fields of Structural Elements According to Three-Dimensional Plans

Characteristic examples of elements of structures under three-dimensional conditions include concrete columns leading in height during the period of construction, the height of the projection portion of which is comparable to the plan dimensions.

The method of solution of three-dimensional problems differs little in principle from the method of solution of two-dimensional problems: basically, multiple finite integral transforms are used with respect to coordinates, plus Laplace transforms with respect to time and the Green function. The solutions are usually rather cumbersome. In this section, we will present only one solution of a heat conductivity problem for a semilimited column.

The problem can be formulated as

$$\frac{\partial T}{\partial \tau}=a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \quad (0<x<R,\ 0<y<L,\ 0<z<D,\ \tau>0); \tag{4-65}$$

$$\begin{gathered}
T(x,\ y,\ z,\ \tau)=T_0 \quad (0\leqslant x\leqslant R,\ 0\leqslant y\leqslant L,\ 0\leqslant z\leqslant D);\\
\frac{\partial T((i-1)R,\ y,\ z,\ \tau)}{\partial x}=(-1)^i h_i[T_1-T((i-1)R,\ y,\ z,\ \tau)] \quad (i=1,\ 2);\\
\frac{\partial T(x,\ (j-3)L,\ z,\ \tau)}{\partial y}=(-1)^j h_j[T_1-T(x,\ (j-3)L,\ z,\ \tau)] \quad (j=3,\ 4);\\
\frac{\partial T(x,\ y,\ 0,\ \tau)}{\partial z}=-h_5[T_1-T(x,\ y,\ 0,\ \tau)];\\
\frac{\partial T(x,\ y,\ \infty,\ \tau)}{\partial z}=0,\ T(x,\ y,\ \infty,\ \tau)\neq\infty.
\end{gathered} \tag{4-66}$$

Let us assume

$$T(x,\ y,\ z,\ \tau)=T_1-\vartheta(x,\ y,\ z,\ \tau).$$

Then for the function $\theta(x,\ y,\ z,\ \tau)$ we have differential equation (4-65), the initial condition

$$\vartheta(x,\ y,\ z,\ 0)=T_1-T_0$$

and homogeneous boundary conditions such as (4-66).

Based on the property of multiplication of solutions, we can write:

$$\theta(x, y, z, \tau) = \theta_1(x, \tau)\theta_2(y, \tau)\theta_3(z, \tau),$$

where $\theta_1(x, \tau)$ is the solution of the one-dimensional problem

$$\frac{\partial\theta_1}{\partial\tau} = a\frac{\partial^2\theta_1}{\partial x^2} \quad (0 < x < R,\ \tau > 0);$$
$$\theta_1(x, 0) = T_1 - T_0 \quad (0 \leqslant x \leqslant R);$$
$$\frac{\partial\theta_1((i-1)R, \tau)}{\partial x} = (-1)^{i+1}h_i\theta_1((i-1)R, \tau) \quad (i = 1, 2);$$

$\theta_2(y, \tau)$ is the solution of the problem

$$\frac{\partial\theta_2}{\partial\tau} = a\frac{\partial^2\theta_2}{\partial y^2} \quad (0 < y < L,\ \tau > 0);$$
$$\theta_2(y, 0) = 1 \quad (0 \leqslant y \leqslant L);$$
$$\frac{\partial\theta_2((j-3)L, \tau)}{\partial y} = (-1)^{j+1}h_j\theta_2((j-3)L, \tau) \quad (j = 3, 4);$$

$\theta_3(z, \tau)$ is the solution of the problem

$$\frac{\partial\theta_3}{\partial\tau} = a\frac{\partial^2\theta_3}{\partial z^2} \quad (0 < z < \infty,\ \tau > 0);$$
$$\theta_3(z, 0) = 1 \quad (0 \leqslant z < \infty);$$
$$\frac{\partial\theta_3(0, \tau)}{\partial z} = -h_5\theta_3(0, \tau); \quad \frac{\partial\theta_3(\infty, \tau)}{\partial z} = 0; \quad T(\infty, \tau) \neq \infty.$$

These solutions can be easily written, using the results presented earlier.

Finally,

$$T(x, y, z, \tau) = T_1 - \theta_1(x, \tau)\theta_2(y, \tau)\theta_3(z, \tau). \tag{4-67}$$

4-5. Calculations of the Temperatures of Internal Zones of a Concrete Mass

In those cases when special measures are not used to regulate the temperature of a concrete mass (for example during construction of concrete masses without pipe cooling), the internal zones of sufficiently high blocks (3 m in height or more) are under near-adiabatic conditions for long periods of time.

In order to determine the temperature of these zones, we present below calculation formulas which consider certain characteristic forms of the heat liberation intensity functions.

1. The generalized heat liberation intensity function

$$q(\tau, T) = q_\nu (d_\nu + b_\nu T)\, e^{-m_\nu \tau}. \tag{4-68}$$

Here q_ν, d_ν, b_ν, m_ν are piecewise-constant functions of time, defined in the interval $(\tau_{\nu-1}, \tau_\nu)$ $(\nu = 1, 2, \ldots, s)$.

If the parameter q_ν, d_ν, b_ν, m_ν are known, the temperature of an adiabatically isolated volume of concrete, the initial temperature of which is T_0, is calculated from the formula

$$T = T_0 + \exp\left[\sum_{\nu=1}^{s} c_\nu\right] \sum_{\nu=1}^{s} \left(\frac{d_\nu}{b_\nu} + T_0\right) (\exp[2\rho_\nu \operatorname{sh}(m_\nu \tau_-)] - 1), \tag{4-69}$$

where

$$c_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma} (\exp[-m_\nu \tau_{\nu-1}] - \exp[-m_\nu \tau_\nu]);$$

$$\rho_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_+}; \quad \tau_+ = \frac{\tau_\nu + \tau_{\nu-1}}{2}; \quad \tau_- = \frac{\tau_\nu - \tau_{\nu-1}}{2}.$$

This formula can be produced from expression (4-10) for the temperature of a concrete wall if we assume:

$$n = 1; \ \mu_n = 0; \ A_n = 1; \ T_c = T_0.$$

The integral thus produced

$$\psi_{\nu n} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu}\right] \int_{\tau_{\nu-1}}^{\tau_\nu} \exp\left[-m_\nu \zeta + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta}\right] d\zeta,$$

as we can easily see is equal to

$$\psi_{\nu n} = \frac{c\gamma}{q_\nu b_\nu} (\exp[2\rho_\nu \operatorname{sh}(m_\nu \tau_-)] - 1).$$

Note. Formula (4-69) can be used for all values of τ for which the parameters q_ν, d_ν, b_ν, m_ν are fixed. Extrapolation to higher values of τ is not recommended.

2. The heat liberation intensity function of I. D. Zaporozhets

$$q(\tau, T) = \frac{Q_{\text{макс}}}{m-1} A_{20} 2^{\frac{T-20}{\varepsilon}} \left[1 + A_{20} \int_0^\tau 2^{\frac{T-20}{\varepsilon}} d\tau\right]^{-\frac{m}{m-1}}, \tag{4-70}$$

where Q_{max}, A_{20}, ε, m are parameters (according to the recommendations of I. D. Zaporozhets, we can assume $m = 2.2$, $\varepsilon = 10$).

In calculating the temperature of an adiabatically isolated volume, we should use the relationships which follow from the regularity of times of equal heat liberation.

If we know the curve of adiabatic heat liberation of concrete of a given composition with initial temperature of concrete mixture T_{01}, the curve of adiabatic heat liberation of the concrete of the same composition, but with initial temperature of concrete mixture T_{02} is constructed on the basis of the relationship

$$\frac{\tau_1}{\tau_2} = 2^{\frac{T_{01} - T_{02}}{\varepsilon}}, \tag{4-71}$$

where τ_1 and τ_2 are the corresponding times of equal heat liberation.

If we know the curve of isothermal heat liberation with constant temperature T_{is}, the times of equal adiabatic heat liberation T_{ad} with the initial temperature of the concrete mixture T_0 are calculated by the formulas

$$\tau_{ad} = 2^{\frac{T_{is} - T_0}{\epsilon}} \int_0^{\tau_{is}} \exp\left[-\frac{Q_{is}(\tau)\ln 2}{\epsilon c\gamma}\right] d\tau. \qquad (4\text{-}72)$$

Using the substitution

$$z(\tau_{is}) = \exp\left[\frac{\ln 2}{\epsilon c\gamma} Q_{max} (1 + A_{T_{is}}\tau)^{-\frac{1}{m-1}}\right]$$

this formula is converted to

$$\tau_{ad} = 2^{\frac{T_{is} - T_0}{\epsilon}} \frac{(m-1)\left[\frac{\ln 2}{\epsilon c\gamma} Q_{max}\right]^{m-1}}{A_{T_{is}} \exp\left[\frac{\ln 2}{\epsilon c\gamma} Q_{max}\right]} \int_{z(\tau_{is})}^{z(0)} \ln^{-m} z\, dz.$$

For m = 2.2, this last integral was tabulated by I. D. Zaporozhets [50].

Based on the relationships presented, we can construct the curve of adiabatic heat liberation $Q_{ad}(\tau)$. The temperature of an adiabatically insulated volume T_{ad} is calculated on the basis of this curve from the formula

$$T_{ad} = T_0 + \frac{Q_{ad}}{c\gamma}.$$

3. The heat liberation intensity function depends only on time $q(\tau)$.

If $q(\tau)$ is approximated by sectors of the exponents

$$q(\tau) = q_\nu e^{-m_\nu \tau}, \qquad (4\text{-}73)$$

where q_ν, m_ν are piecewise-constant functions of time, defined over $(\tau_{\nu-1}, \tau_\nu)$ (ν = 1, 2,..., s), then

$$T = T_0 + \sum_{\nu=1}^{s} \frac{q_\nu}{m_\nu c\gamma} \left(e^{-m_\nu \tau_{\nu-1}} - e^{-m_\nu \tau_\nu}\right). \tag{4-74}$$

If q(τ) is approximated by the sum of exponential functions

$$q(\tau) = \sum_{\nu=1}^{i} q_\nu e^{-\nu m \tau}, \tag{4-75}$$

then

$$T = T_0 + \sum_{\nu=1}^{i} \frac{q_\nu}{\nu m c\gamma} \left(1 - e^{-\nu m_\nu \tau}\right). \tag{4-76}$$

CHAPTER 5. METHODS OF CALCULATION OF TEMPERATURE FIELDS IN CONCRETE MASSES AS THEY ARE CONSTRUCTED

5-1. Statement of the Basic Problems

Pouring of concrete during construction of hydraulic engineering structures is performed in blocks, the height of which varies from 0.5 to 3-6 m and even more (15-20 m).

In American practice, the height of blocks is generally 1.52 m (Hoover, Shasta, Hungry Horse and other dams), though in many cases it is 2.28 m (Glen Canyon, Table Rock and other dams). In Europe, block heights are about 3 m (Grand Diksans Dam, Switzerland -- 3.2 m). In Canada, dams are commonly constructed by the "high block" method, with block heights as great as 20 m.

The construction of the Bratsk Hydroelectric Power Plant Dam was performed in blocks of 1.5 m height in the zone near the bottom, increasing to 3-6 m at the remaining levels. The block heights of the Krasnoyarsk Hydroelectric Power Plant Dam were similar to these dimensions. The height of blocks of the Toktogul'skaya Hydroelectric Power Plant was 0.5, 0.75 and 1.0 m. In construction norms SN 123-60 [82], it is recommended that when the column method is used, block lengths of 9 to 20 m and heights of 1.5 to 3.0 m be used, while for long blocks (see below) heights of 0.75 to 1.5 m are recommended.

The difference in block heights in the base and upper zones of a mass (from 0.75-1.5 m to 1.5-3.0 m) is typical for all hydraulic engineering structures, both domestic and foreign. Furthermore, the blocks of a mass frequently have different heights for reasons of technological and production convenience.

The blocks of a dam are ordinarily poured in layers 30-50 cm high[1], layer being poured over layer within the course of a few hours. Interruptions in pouring of blocks depend on their height, the rates of construction used and other factors, and vary from 3-5 to 10 days or more.

Due to this great difference in the intervals of coverage of one layer by the next and one block by the next, it can be assumed that the blocks are poured instantly, if their height is not too great. For high blocks it is

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[1]One exception is the blocks constructed by the Toktogul'skaya method; here the layer height was equal to the block height -- 0.5, 0.75 or 1.0 m.

more correct to assume continuous growth of the height of the block. As is shown in § 5-3, blocks up to 3-6 m can be considered to be in the former group.

The separation of blocks into ordinary and high blocks with the use of models of instantaneous and extended, but continuous growth in height allows an economic approach, from the standpoint of cost of calculation, to the calculation of temperature fields in concrete masses, while retaining accuracy sufficient for engineering practice.

The plan dimensions of masses depend on the system used to divide the structure into concreting blocks.

With column-type division, typical plan dimensions are 7.63 x 9.15 m, 15.2 x 18.3 m (Hoover Dam), 15.2 x 15.2 m (Shasta Dam), 22 x 13.8 m, 15.0 x 13.8 m (Bratsk Dam), 11.5 x 7.5 (9.0; 11.0; 15.0) m (Krasnoyarsk Dam), etc.

In the mid-1940's in the USA, then in other countries as well, structures began to be divided into long blocks, encompassing entire dam sections, though the sections are sometimes divided by a longitudinal seam into two blocks. The characteristic dimensions of long blocks are: 15 x 108 (105) m (Detroit and Pine Flat Dams, USA), (11-18) x 95 (100) m (Sar'yar and Kemer Dams, Turkey), (15-18) x 80 m (Bor Dam, France), 20-57 m (Compo-Fero Dam, Italy), 15 x 70 m (Krasnoyarsk Dam), 16 x 60 m, 32 x 60 m, 32 x 135 m (Toktogul'skaya Dam).

Long blocks also include column blocks in which one of the plan dimensions is 1.8-2 times or more greater than the other. They include, for example, the blocks of the Glen Canyon Dam, USA (16 x 35-40 m, 18.3 x 64.0 m), the Grand Diksans Dam (16 x 32-56 m), Tokotura Dam in Japan (15 x 37 m), etc.

As was noted in Chapter 1, depending on the relationship of plan dimensions and the height of the masses being studied and the purpose of the investigation, we make a distinction among three-dimensional (spatial), two-dimensional (planar) and one-dimension (linear) temperature problems.

Spatial problems adequately describe the thermal phenomena in masses of any size. However, due to the large volume of computation, their use is desirable only for the description of the thermal state of masses, the height of which is comparable to the plan dimensions. This obtains, for example, for the construction of columns with a significant difference in height between neighboring masses, in the construction of broad cooling seams, facilitating intensive heat exchange from the side surfaces of the column, for free-standing columns and high blocks, etc.

Planar (two-dimensional) problems are suitable for determination of the temperature fields in masses, one of the dimensions of which is significantly (by 4 times or more) greater than the other two. Vertical cross sections of masses, erected in long blocks, vertical transverse sections of complete (and incomplete) profile of gravity and arch dams, placed at levels in the horizontal cross section of rather high masses at some distance from the

base, etc. form a very incomplete list of examples of planar areas.

Finally, linear (one-dimensional) problems yield the temperature fields in the middle portions of masses, two dimensions of which are greater than the third (stratified concreting blocks, masses poured by the Toktogul'skaya method, etc.). They can also be used to estimate the maximum temperatures along the axis of columnar blocks.

Two-dimensional and one-dimensional problems approximately describe the picture of the phenomena. Therefore, with this statement of the problem we must consider the heat losses through those faces of the mass which are not included in the edge conditions of the problem.

The temperature of a concrete mixture and, consequently, the initial temperature of the blocks, does not remain constant in the process of construction of a mass; it varies from block to block and it should be kept in mind in formulating and solving the corresponding thermal problems.

It is equally important to consider the initial distribution of temperature in the base (rock, old concrete), where the thermal prehistory plays a significant role. Therefore, the calculation method should include the possibility of analysis of the thermal mode of the base before the beginning of pouring of the mass.

Heat exchange with the horizontal and lateral surfaces of masses basically occurs by convection (boundary condition of third kind). The heat transfer coefficient here, as a rule, should be taken to mean its effective value. The possibility of this approach, simplifying consideration of the thermal protective properties of the decking, was noted in § 2-3. At the surfaces of contact between concrete and water (when water is poured over the horizontal surfaces of a mass during interruptions between pouring of blocks, in case of artificial creation of a water "jacket" on vertical surfaces, when some water flows through the area during construction, and the head and tail water of the dam as the reservoir is filled, during the period of operation of the dam, etc.), boundary conditions of the first kind can be used.

During relatively brief interruptions between pouring of blocks, when concreting is performed beneath a tent, etc., the air temperature can be assumed constant. In the general case, the air temperature, like the water temperature, should be assumed to be a function of time, fixed either analytically or in tabular form (for more detail see § 2-3).

As was indicated in § 2-1, the heat-physical characteristics of the concrete can be assumed to be independent of temperature and time. However, one should keep in mind the difference between heat-physical characteristics of concrete and of the rock base, particularly when the thermal state in question is that near the contact zone.

Heat liberation in the concrete is one of the primary factors determining the thermal mode of the mass (see § 2-2).

For a more complete and proper description of the temperature fields of concrete masses, we must apply both analytic and finite-difference methods of calculation in combination. This allows us to make effective use of modern electronic computers for our calculations and regulation of the temperature mode of water engineering structures.

The methods of calculation of temperature fields of concrete masses in the process of construction have been developed in the studies of A. V. Belov and P. I. Vasil'yev [15, 18, 22], G. N. Danilova and N. A. Buchko [14, 39, 40], Sh. N. Plyat [98, 100, 102-106], Sh. N. Plyat and L. B. Sapozhnikov [96, 97], G. I. Chilingarishvili [133-135], V. M. Shteynberg, I. Ye. Prokopovich and I. V. Gol'dfarb [140-142], I. Babushka and L. Meyzlik [151, 152], R. Glover [156], R. Karlson [153] etc. [149, 172].

5-2. Analytic Methods of Calculation of Temperature Fields in Concrete Masses Growing by Blocks

Spatial Temperature Field

A study is made of the temperature field of a concrete column. Beginning at a certain moment in time, the column begins to grow in height in discrete blocks. The initial temperature of the base is variable. The blocks are placed instantaneously, each block having its own constant initial temperature.

After placement, the process of heat liberation begins in the block, the intensity of which depends exponentially on time. Interruptions in placement of blocks vary. The temperature and heat exchange coefficients on the horizontal and vertical surfaces also differ. The heat field characteristics of the column are constant.

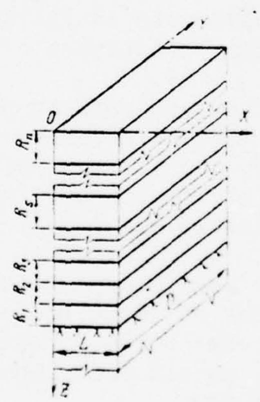

Figure 5-1. Diagram of Calculation Area in Problem of Spatial Temperature Field of a Concrete Mass

Suppose at the given stage of growth a column consists of a base and $\bar{n}$ blocks. Let us place the coordinate origin on the upper horizontal surface of block $s = \bar{n}$ and direct the OZ axis into the depth of the mass, the directions of the OY and OX axis being as usual (Figure 5-1). Then the mathematical problem can be formulated as follows.

The system of differential equations

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a\left[\frac{\partial^2 T_0}{\partial z^2} + \frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2}\right]$$

$$\left(\sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty,\ -L < x < L,\ -D < y < D,\ \tau > 0\right);$$

$$\frac{\partial T_s}{\partial \tau^{(\bar{n})}} = a\left[\frac{\partial^2 T_s}{\partial z^2} + \frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2}\right] + \frac{q_0}{c\gamma} e^{-mt_s}$$

$$\left(\sum_{j=s+1}^{\bar{n}} R_j < z_{\bar{n}} < \sum_{j=s}^{\bar{n}} R_j,\ -L < x < L,\ -D < y < D,\ \tau > 0,\right.$$

$$\left. s = 1, \ldots, \bar{n}\right). \tag{5-1}$$

The initial conditions

$$T_0(z_{\bar{n}}, x, y, 0] = T_0^{\text{ск}} + \Phi_0^{(\bar{n}-1)}(z_{\bar{n}}, x, y, \tau_{\bar{n}});$$

$$T_l(z_{\bar{n}}, x, y, 0) = T_2 + \Phi_{e\,l}^{(\bar{n}-1)}(z_{\bar{n}}, x, y, \tau_{\bar{n}}) \quad (l = 1, 2, \ldots, \bar{n}-1);$$

$$T_{\bar{n}}(z_{\bar{n}}, x, y, 0) = T^{(\bar{n})}. \tag{5-2}$$

The boundary conditions on the surfaces of the concrete column

$$\frac{\partial T_{\bar{n}}(0, x, y, \tau^{(\bar{n})})}{\partial z_{\bar{n}}} = -h_{z(\bar{n})}[T_1 - T_{\bar{n}}(0, x, y, \tau^{(\bar{n})})];$$

$$\frac{\partial T_s(z_{\bar{n}}, L, y, \tau^{(\bar{n})})}{\partial x} = h_{x(\bar{n})}[T_2 - T_s(z_{\bar{n}}, L, y, \tau^{(\bar{n})})];$$

$$\frac{\partial T_s(z_{\bar{n}}, 0, y, \tau^{(\bar{n})})}{\partial x}=0 \text{ (condition of symmetry);}$$

$$\frac{\partial T_s(z_{\bar{n}}, x, D, \tau^{(\bar{n})})}{\partial y}=h_{y(\bar{n})}[T_2 - T_s(z_{\bar{n}}, x, D, \tau^{(\bar{n})})];$$

$$\frac{\partial T_s(z_{\bar{n}}, x, 0, \tau^{(\bar{n})})}{\partial y}=0 \text{ (condition of symmetry)} \quad (5\text{-}3)$$

and on the surfaces of the base

$$\frac{\partial T_0(\infty, x, y, \tau^{(\bar{n})})}{\partial z_{\bar{n}}}=0;\ T_0(\infty, x, y, \tau^{(\bar{n})})\neq\infty;$$

$$\frac{\partial T_0(z_{\bar{n}}, L, y, \tau^{(\bar{n})})}{\partial x}=h_{x(\bar{n})}[T^{\text{ск}} - T_0(z_{\bar{n}}, L, y, \tau^{(\bar{n})})];$$

$$\frac{\partial T_0(z_{\bar{n}}, x, D, \tau^{(\bar{n})})}{\partial y}=h_{y(\bar{n})}[T^{\text{ск}} - T_0(z_{\bar{n}}, x, D, \tau^{(\bar{n})})];$$

$$\frac{\partial T_0(z_{\bar{n}}, 0, y, \tau^{(\bar{n})})}{\partial x}=0 \quad \text{(condition of symmetry);}$$

$$\frac{\partial T_0(z_{\bar{n}}, x, 0, \tau^{(\bar{n})})}{\partial y}=0 \quad \text{(condition of symmetry).} \quad (5\text{-}4)$$

The conditions of conjugation at "block-block" and "block-base" boundaries are

$$T_s\left(\sum_{j=s}^{\bar{n}} R_j, x, y, \tau^{(\bar{n})}\right)=T_{s-1}\left(\sum_{j=s}^{\bar{n}} R_j, x, y, \tau^{(\bar{n})}\right);$$

$$\frac{\partial T_s\left(\sum_{j=s}^{\bar{n}} R_j, x, y, \tau^{(\bar{n})}\right)}{\partial z_{\bar{n}}}=\frac{\partial T_{s-1}\left(\sum_{j=s}^{\bar{n}} R_j, x, y, \tau^{(\bar{n})}\right)}{\partial z_{\bar{n}}}. \quad (5\text{-}5)$$

Here and below, the following symbols are used:

n -- the maximum number of concrete blocks in the mass;
$\bar{n}$ -- the number of blocks in the mass at the growth stage ($\bar{n} = 1, 2, \ldots, n$);
s -- the number of blocks (counting from the base, for which $s = 0$) ($s = 1, 2, \ldots, \bar{n}$);
$2L \times 2D$ -- the plan dimensions of the mass;
T -- the temperature, T_s (T_ℓ) and T_0 -- the temperature of the sth (ℓth) block and the base at any moment in time;
$T^{(s)}$ -- the temperature of the sth block at the moment of placement (constant);
T^{CK} -- the constant component of the temperature of the base, as well as the ambient temperature on the vertical surfaces in the area of the base;
T_1 and T_2 -- the ambient temperature on the horizontal and vertical surfaces in the area of the concrete portion of the mass as the next, $\bar{n}$th, block is placed;
τ -- time, $\tau^{(\bar{n})}$ -- the time from the moment of placement of the last, $\bar{n}$th block;
$\tau_{\bar{n}}$ -- the time interval between placement of the ($\bar{n}$ - 1)th block and the $\bar{n}$th block which covers it;
t_s -- the "life" time of the sth block;
z, x, y -- coordinates, $z_{\bar{n}}$ -- the coordinate for a mass consisting of $\bar{n}$ blocks and the base;
$h_{x(\bar{n})}$, $h_{y(\bar{n})}$, $h_{z(\bar{n})}$ -- the relative heat transfer coefficients for the mass in the stage of growth.

The solution of the problem was produced by a double finite integral transform with respect to the coordinates, a Laplace transform with respect to time and the Green function. Cumbersome intermediate computations are not presented.

The final result was

$$T_0(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) = T^{CK} + \Phi_0^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})})$$
$$\left(\sum_{j=1}^{\bar{n}} R_j \leq z_{\bar{n}} < \infty, \ -L \leq x \leq L, \ -D \leq y \leq D, \ \tau > 0\right);$$

$$T_s(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) = T_2 + \Phi_c^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})})$$
$$\left(\sum_{j=s+1}^{\bar{n}} R_j \leq z_{\bar{n}} \leq \sum_{j=s}^{\bar{n}} R_j, \ -L \leq x \leq L, \ -D \leq y \leq D, \ \tau > 0; \ s = 1, 2, \ldots, \bar{n}\right). \tag{5-6}$$

where

$$\Phi_i^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) = F^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) + f_i^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) +$$

$$+ \frac{4}{LD}\sum_{p=1}^{\infty}\sum_{r=1}^{\infty}(\mathrm{Bi}_x^2 + \mathrm{Bi}_x + \mu_p^2)^{-1}(\mathrm{Bi}_y^2 + \mathrm{Bi}_y + \varkappa_r^2)^{-1} \times$$

$$\times (\mathrm{Bi}_x^2 + \mu_p^2)(\mathrm{Bi}_y^2 + \varkappa_r^2)\exp[-aK_{pr}\tau^{(\bar{n})}] \times$$

$$\times \int_{R_{\bar{n}}}^{\infty}\int_0^L\int_0^D [\Phi^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, y, \tau_{\bar{n}})]\cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D} \times$$

$$\times \left\{\frac{1}{2\sqrt{\pi a\tau^{(\bar{n})}}}\left(\exp\left[-\frac{(z_{\bar{n}} - \xi)^2}{4a\tau^{(\bar{n})}}\right] + \exp\left[-\frac{(z_{\bar{n}} + \xi)^2}{4a\tau^{(\bar{n})}}\right]\right) -\right.$$

$$- h_{z(\bar{n})}\exp[h_{z(\bar{n})}^2 a\tau^{(\bar{n})} + h_{z(\bar{n})}(z_{\bar{n}} + \xi)]\,\mathrm{erfc}\left[\frac{z_{\bar{n}} + \xi}{2\sqrt{a\tau^{(\bar{n})}}} +\right.$$

$$\left.\left.+ h_{z(\bar{n})}\sqrt{a\tau^{(\bar{n})}}\right]\right\} d\xi\, dx\, dy \quad (i = 0, c);$$

(5-7)

$$\Phi^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, y, \tau_{\bar{n}}) = \begin{cases} \Phi_0^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, y, \tau_{\bar{n}}) & \left(\sum\limits_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty\right); \\ \Phi_c^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, y, \tau_{\bar{n}}) & \left(R_{\bar{n}} < z_{\bar{n}} < \sum\limits_{j=1}^{\bar{n}} R_j\right); \end{cases}$$

$$K_{pr} = \frac{\mu_p^2}{L^2} + \frac{\varkappa_r^2}{D^2};$$

$$f_0^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) = \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_p B_r \cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D} \times$$

$$\times \left[T^{(\bar{n})} - T_2 + \frac{q_0}{c\gamma(m - aK_{pr})}\right]\exp[-aK_{pr}\tau^{(\bar{n})}] - \vartheta(z_{\bar{n}}, x, y, \tau^{(\bar{n})});$$

$$f_2^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})}) = \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_p B_r \cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D} \times$$

$$\times \left\{(T^{(\bar{n})} - T_2)\exp[-aK_{pr}\tau^{(\bar{n})}] + \frac{q_0}{c\gamma(m - aK_{pr})} \times\right.$$

$$\left.\times \left(\exp[-aK_{pr}\tau^{(\bar{n})}] - \exp\left[-m\left(\sum_{j=\nu-1}^{\bar{n}}\tau_j + \tau^{(\bar{n})}\right)\right]\right)\right\} +$$

$$+ \vartheta(z_{\bar{n}}, x, y, \tau^{(\bar{n})});$$

$$\theta(z_{\bar n}, x, y, \tau^{(\bar n)}) = \frac{1}{4}(T^{\text{ck}} - T_2)\sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_p B_r \cos\mu_p \frac{x}{L} \times$$

$$\times \cos\varkappa_r \frac{y}{D}\left\{\exp\left[-\sqrt{K_{pr}}\left(\sum_{j=1}^{\bar n} R_j - z_{\bar n}\right)\right]\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar n} R_j - z_{\bar n}}{2\sqrt{a\tau^{(\bar n)}}} -\right.\right.$$

$$\left. - \sqrt{aK_{pr}\tau^{(\bar n)}}\right] + \exp\left[\sqrt{K_{pr}}\left(\sum_{j=1}^{\bar n} R_j - z_{\bar n}\right)\right]\times$$

$$\left.\times \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar n} R_j - z_{\bar n}}{2\sqrt{a\tau^{(\bar n)}}} + \sqrt{aK_{pr}\tau^{(\bar n)}}\right]\right\};$$

$$F^{(\bar n)}(z_{\bar n}, x, y, \tau^{(\bar n)}) = \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_p B_r \cos\mu_p \frac{x}{L} \cos\varkappa_r \frac{y}{D} W_{pr}(z_{\bar n}, \tau^{(\bar n)});$$

$$W_{pr}(z_{\bar n}, \tau^{(\bar n)}) = \sum_{\rho=1}^{4} \exp[-aK_{pr}\tau^{(\bar n)}] \times$$

$$\times\left\{\sum_{i=1}^{\bar n} a_{\rho i}\operatorname{erfc}\left[\frac{N_\delta(z_{\bar n})}{2\sqrt{a\tau^{(\bar n)}}}\right] + b_{\rho i}\exp[(h^2_{z(\bar n)} + K_{pr})a\tau^{(\bar n)} +\right.$$

$$+ h_{z(\bar n)} N_\gamma(z_{\bar n})]\operatorname{erfc}\left[\frac{N_\gamma(z_{\bar n})}{2\sqrt{a\tau^{(\bar n)}}} + h_{z(\bar n)}\sqrt{a\tau^{(\bar n)}}\right] +$$

$$+ d_{\rho i} I_{1/2}\left(m - aK_{pr}, -\left(\frac{N_\gamma(z_{\bar n})}{2\sqrt{a}}\right)^2, \tau^{(\bar n)}\right) +$$

$$+ g_{\rho i} I_{3/2}\left(m - aK_{pr}, -\left(\frac{N_\delta(z_{\bar n})}{2\sqrt{a}}\right)^2, \tau^{(\bar n)}\right) +$$

$$+ c_\rho \exp\left[(-1)^{\rho-1}\sqrt{K_{pr}}\left(z_{\bar n} + \delta_\rho \sum_{j=1}^{\bar n} R_j\right)\right] \times$$

$$\left.\times \operatorname{erfc}\left[\frac{z_{\bar n} + \delta_\rho \sum_{j=1}^{\bar n} R_j}{2\sqrt{a\tau^{(\bar n)}}} + (-1)^{\rho-1}\sqrt{aK_{pr}\tau^{(\bar n)}}\right]\right\};$$

$$N_\gamma(z_{\bar n}) = \sum_{j=1}^{\bar n} R_j - \gamma_\rho \sum_{j=1}^{i} R_j + z_{\bar n};$$

$$N_\delta(z_{\bar{n}}) = \sum_{j=1}^{\bar{n}} R_j - \delta_\rho \sum_{j=1}^{i} R_j + (-1)^{\rho-1} z_{\bar{n}};$$

$$A_p = (-1)^{p+1} \frac{2\mathrm{Bi}_x}{\mu_p} (\mathrm{Bi}_x^2 + \mathrm{Bi}_x + \mu_p^2)^{-1} \sqrt{\mathrm{Bi}_x^2 + \mu_p^2};$$

$$B_r = (-1)^{r+1} \frac{2\mathrm{Bi}_y}{\varkappa_r} (\mathrm{Bi}_y^2 + \mathrm{Bi}_y + \varkappa_r^2)^{-1} \sqrt{\mathrm{Bi}_y^2 + \varkappa_r^2};$$

$$\delta_\rho = \begin{cases} 1 \text{ where } \rho \leq 2; \\ 0 \text{ where } \rho > 2; \end{cases} \quad \gamma_\rho = \begin{cases} 1 \text{ where } \rho = 1; \\ 0 \text{ where } \rho > 1; \end{cases}$$

$$\mathrm{Bi}_x = h_{x(\bar{n})} L; \quad \mathrm{Bi}_y = h_{y(\bar{n})} D;$$

μ_p and κ_r are the roots of the characteristic equations

$$\cot \mu_p = \frac{\mu_p}{\mathrm{Bi}_x}; \quad \cot \varkappa_r = \frac{\varkappa_r}{\mathrm{Bi}_y}.$$

The algorithms for calculation of the function erfc x and the integral

$$I_{(2k+1)/2}\left(m - aK_{pr}, -\left(\frac{N}{2\sqrt{a}}\right)^2, \tau^{(\bar{n})}\right) =$$
$$= \left(\frac{N}{2\sqrt{a}}\right)^k \int_0^\tau \vartheta^{-\frac{2k+1}{2}} \exp\left[(m - aK_{pr})(\vartheta - \tau^{(\bar{n})}) - \frac{N^2}{4a\vartheta}\right] d\vartheta \quad (k = 0, 1)$$

are presented below. The coefficients $a_{\rho i}$, $b_{\rho i}$, $d_{\rho i}$, $g_{\rho i}$ and c_ρ are determined by expressions presented in Table 5-1.

In defining the function $\Phi^{(\bar{n})}(z_{\bar{n}}, x, y, \tau^{(\bar{n})})$, the essential aspect is the calculation of the integral included in (5-7). Its integrand contains the value of $\Phi^{(\bar{n}-1)}$ for the mass of $(\bar{n} - 1)$ blocks at the moment it is covered with the $\bar{n}$th block.

Let us represent:

$$\int\limits_{R_{\bar{n}}}^{\infty}\int\limits_{0}^{L}\int\limits_{0}^{D}\left[\Phi^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, y, \tau^{(\bar{n})})\right]\cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D}\left\{\quad\right\}d\xi\,dx\,dy =$$

$$= \int\limits_{R_{\bar{n}}}^{\sum_{j=1}^{\bar{n}}(R_j+R_{-j})}\int\limits_{0}^{L}\int\limits_{0}^{D} + \int\limits_{\sum_{j=1}^{\bar{n}}(R_j+R_{-j})}^{\infty}\int\limits_{0}^{L}\int\limits_{0}^{D}\left[\Phi^{(\bar{n}-1)}\right]\cos\mu_p\frac{x}{L}\times$$

$$\times\cos\varkappa_r\frac{y}{D}\left\{\quad\right\}d\xi\,dx\,dy.$$

Here R_{-j} is the height of the "imaginary" block, which in a certain sense is a reflection in the base of the actual block of height R_j. In this case, R_{-j} is a multiple of R_j.

Since with high values of $z_{\bar{n}}$, function $\Phi^{(\bar{n}-1)}$ approaches 0, we can always select a multiplicity factor such that the second integral of the last expression also approaches 0.

Calculation of the first integral is simplified if within the limits of each block, real and "imaginary," function $\Phi^{(\bar{n}-1)}$ is approximated by a formula such as

$$\Phi_j^{(\bar{n}-1)} = P_\alpha(z_{\bar{n}})\,P_\beta(x)\,P_\gamma(y),$$

where P_α, P_β and P_γ are polynomials of power α, β and γ.

The power of a polynomial is determined from the desired accuracy. All calculations in this case are reduced to calculations using simple recurrent formulas.

Two-Dimensional (Planar) Temperature Field

The statement of the planar problem is basically the same as that of the spatial problem. The system of differential equations

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T_0}{\partial z_{\bar{n}}^2} + \frac{\partial^2 T_0}{\partial x^2}\right) - H_{\bar{n}}(T_0 - T^{CK})$$
$$\left(\sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty, -L < x < L, \tau > 0\right);$$

$$\frac{\partial T_s}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T_s}{\partial z_{\bar{n}}^2} + \frac{\partial^2 T_s}{\partial x^2}\right) - H_{\bar{n}}(T_s - T_3) + \frac{q_0}{c\gamma} e^{-mt_s}$$
$$\left(\sum_{j=s+1}^{\bar{n}} R_j < z_{\bar{n}} < \sum_{j=s}^{\bar{n}} R_j, -L < x < L, \tau > 0;\ s = 1, 2, \ldots, \bar{n}\right).$$

(5-8)

The terms $H_{\bar{n}}(T_0 - T^{CK})$ and $H_{\bar{n}}(T_s - T_3)$ are introduced to the differential equations to consider the heat losses from the third (along the OY axis) dimension. Physically, this means that the heat losses are looked upon as negative heat sources distributed through the volume.

Parameter $H_{\bar{n}}$ is taken identical for all bases, as well as for the mass. The ambient temperature is equal to T^{CK} in the base and T_3 in the area of the lateral surface of the concrete portion of the column.

The edge conditions of the planar problem in question do not differ in principle from the edge conditions of the three-dimensional problem. It should be kept in mind that the problem is two-dimensional, i.e., we must consider that all functions depend on the coordinates z and x.

The solution of the problem is:

$$T_0(z_{\bar{n}}, x, \tau^{(\bar{n})}) = T^{CK} + \Phi_0^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})})$$
$$\left(\sum_{j=1}^{\bar{n}} R_j \leqslant z_{\bar{n}} < \infty, -L \leqslant x \leqslant L, \tau > 0\right);$$

$$T_s(z_{\bar{n}}, x, \tau^{(\bar{n})}) = T_3 + \Phi_s^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})})$$
$$\left(\sum_{j=s+1}^{\bar{n}} R_j \leqslant z_{\bar{n}} \leqslant \sum_{j=s}^{\bar{n}} R_j, -L \leqslant x \leqslant L, \tau > 0;\quad s = 1, 2, \ldots, \bar{n}\right),$$

(5-9)

where

$$\Phi_i^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) = F^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) + f_i^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) +$$

$$+ \frac{2}{L} \sum_{p=1}^{\infty} (\mathrm{Bi}_x^2 + \mathrm{Bi}_x + \mu_p^2)^{-1} (\mathrm{Bi}_x^2 + \mu_p^2) \exp[-aK_{pH}\tau^{(\bar{n})}] \times$$

$$\times \int_{R_{\bar{n}}}^{\infty} \int_0^L [\Phi^{(\bar{n}-1)}(z_n, \xi, x, \tau_{\bar{n}})] \cos \mu_p \frac{x}{L} \left\{ \frac{1}{2\sqrt{a\tau^{(\bar{n})}}} \left(\exp\left[-\frac{(z_{\bar{n}} - \xi)^2}{4a\tau^{(\bar{n})}} \right] + \right.\right.$$

$$\left.\left. + \exp\left[-\frac{(z_{\bar{n}} + \xi)^2}{4a\tau^{(\bar{n})}} \right] \right) - h_{z(\bar{n})} \exp[h_{z(\bar{n})}^2 a\tau^{(\bar{n})} + h_{z(\bar{n})}(z_{\bar{n}} + \xi)] \times$$

$$\times \operatorname{erfc}\left[\frac{z_{\bar{n}} + \xi}{2\sqrt{a\tau^{(\bar{n})}}} + h_{z(\bar{n})} \sqrt{a\tau^{(\bar{n})}} \right] \right\} d\xi\, dx \quad (i = 0, c);$$

$$f_0^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) = -\theta(z_{\bar{n}}, x, \tau^{(\bar{n})});$$

$$f_c^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) = (T_3 - T_2) \left\{ 1 - \left[\mathrm{Bi}_x \operatorname{ch} \sqrt{\frac{H_{\bar{n}} L^2}{a}} + \right.\right.$$

$$\left.\left. + \sqrt{\frac{H_{\bar{n}} L^2}{a}} \operatorname{sh} \sqrt{\frac{H_{\bar{n}} L^2}{a}} \right]^{-1} \mathrm{Bi}_x \operatorname{ch} \sqrt{\frac{H_{\bar{n}} L^2}{a}} \frac{x}{L} \right\} -$$

$$- \frac{q_0}{c\gamma(m - H_{\bar{n}})} \exp\left[-m \left(\sum_{j=2}^{\bar{n}} \tau_j + \tau^{(\bar{n})} \right) \right] \times$$

$$\times \left\{ 1 - \left[\mathrm{Bi}_x \cos \sqrt{\frac{(m - H_{\bar{n}})L^2}{a}} - \sqrt{\frac{(m - H_{\bar{n}})L^2}{a}} \times \right.\right.$$

$$\left.\left. \times \sin \sqrt{\frac{(m - H_{\bar{n}})L^2}{a}} \right]^{-1} \mathrm{Bi}_x \cos \sqrt{\frac{(m - H_{\bar{n}})L^2}{a}} \frac{x}{L} \right\} + \theta(z_{\bar{n}}, x, \tau^{(\bar{n})});$$

$$\theta(z_{\bar{n}}, x, \tau^{(\bar{n})}) = \frac{1}{4} \sum_{p=1}^{\infty} A_p \cos \mu_p \frac{x}{L} \left\{ \left(T^{\text{ск}} - T_2 - \frac{H_{\bar{n}}(T_3 - T_2)}{aK_{pH}} \right) \times \right.$$

$$\times \left(\exp\left[-\left(\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}} \right) \sqrt{K_{pH}} \right) \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}} - \right.\right.$$

$$\left. - \sqrt{aK_{pH}\tau^{(\bar{n})}} \right] + \exp\left[\left(\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}} \right) \sqrt{K_{pH}} \right] \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}} + \right.$$

$$+\sqrt{aK_{pH}\tau^{(\bar{n})}}\Bigg]\Bigg) + 4\left(T^{(\bar{n})} - T_2 + \frac{q_0}{c\gamma(m - aK_{pH})} - \right.$$

$$\left.\frac{H_{\bar{n}}(T_3 - T_2)}{aK_{pH}} \exp\left[-aK_{pH}\tau^{(\bar{n})}\right]\right)\Bigg];$$

$$F^{(\bar{n})}(z_{\bar{n}}, x, \tau^{(\bar{n})}) = \sum_{p=1}^{\infty} A_p \cos\mu_p \frac{x}{L} W_{pH}(z_{\bar{n}}, \tau^{(\bar{n})});$$

$$K_{pH} = \frac{\mu_p^2}{L^2} + \frac{H_{\bar{n}}}{a}. \qquad (5\text{-}10)$$

The value of the coefficients $a_{\rho i}$, $b_{\rho i}$, $d_{\rho i}$, $g_{\rho i}$ and c_ρ are presented in Table 5-1. The remaining functions are determined by expressions similar to the expressions in the case of the three-dimensional problem, if we replace K_{pr} by K_{pH} in them.

Let us discuss in more detail the calculation of the integral of expression (5-10).

Based on the considerations presented earlier, this integral is equal to

$$J = \int_0^L \cos\mu_p \frac{x}{L}\, dx \int_{R_{\bar{n}}}^{\sum_{j=1}^{\bar{n}} (R_j + R_{-j})} [\Phi^{(\bar{n}-1)}(z_{\bar{n}}, \xi, x, \tau_{\bar{n}})]\{\ \}\, d\xi.$$

Within the limits of each block, real and "imaginary", we can approximate function $\Phi^{(\bar{n}-1)}$ by a νth power polynomial

$$\Phi^{(\bar{n}-1)} = \sum_{i=0}^{\nu} \alpha_i(x) z^i.$$

For fixed x_k ($k = 1, 1, \ldots, \nu$), coefficients $\alpha_i(x_k)$ are established as a result of solution of the system of equations produced from the values of $\Phi^{(\bar{n}-1)}$ at ($\nu + 1$) points equally separated along z. Partial integrals

appear, which are calculated from the recurrent formulas (see below).

The set of values of the integral

$$\sum_{j=1}^{\bar{n}} \int_{R_{\bar{n}}}^{(R_j + R_{-j})} [\Phi^{(\bar{n}-1)}]\{\ \}\, d\xi$$

at $(\nu + 1)$ points with respect to x_k we determine a certain function $G(x, \tau_{\bar{n}})$, which we also approximate by a polynomial of rth power

$$G(x, \tau_{\bar{n}}) = \sum_{\rho=0}^{r} \beta_\rho x^\rho.$$

Then

$$J = \sum_{\rho=0}^{r} \left(\frac{L}{\mu_p}\right)^{\rho+1} J_\rho,$$

where

$$J_\rho = \int_0^{\mu_p} \zeta^\rho \cos\zeta \, d\zeta.$$

Here

$$J_{\rho>1} = \mu_p^{\rho-1} (\rho \cos\mu_p + \mu_p \sin\mu_p) - \rho(\rho-1) J_{\rho-2};$$
$$J_0 = \sin\mu_p, \quad J_1 = \cos\mu_p + \mu_p \sin\mu_p - 1.$$

One-Dimensional Temperature Field

The system of differential equations

TABLE 5-1. VALUES OF THE COEFFICIENTS $a_{\rho i}$, $b_{\rho i}$, $d_{\rho i}$, $g_{\rho i}$ and c_{ρ} FOR THE SPATIAL AND PLANAR PROBLEMS

a_{1i}	a_{2i}	a_{3i}	a_{4i}
$-\frac{q_0}{2(m-aK)}\exp\left[-m\sum_{j=i+2}^{\bar{n}}\tau_j\right]\times$ $\times(1-\exp[-m\tau_{i+1}]),\ i\neq\bar{n}$ $T_2-T^{(\bar{n})}+\frac{q_0}{(m-aK)}+$ $+\left\{\frac{H_{\bar{n}}(T_3-T_2)}{aK}\right\},\ i=\bar{n}$	$-a_{1i},\ i\neq\bar{n}$ $0,\ i=\bar{n}$	$\frac{q_0}{2(m-aK)}\times$ $\times\exp\left[-m\sum_{j=2}^{\bar{n}}\tau_j\right]-$ $-\left\{\frac{H_{\bar{n}}(T_3-T_2)}{aK}\right\},\ i=1$ $0,\ i\neq 1$	$-a_{3i}$
b_{1i}	b_{2i}	b_{3i}	b_{4i}
$-\frac{q_0}{(h^2_{z(\bar{n})}a+m-aK)}\times$ $\times\exp\left[-m\sum_{j=i+2}^{\bar{n}}\tau_j\right]\times$ $\times(1-\exp[-m\tau_{i+1}]),\ i\neq\bar{n}$	$\frac{(T^{cK}-T_2)h^2_{z(\bar{n})}}{h^2_{z(\bar{n})}-K}-$ $-\frac{q_0}{h^2_{z(\bar{n})}a+m-aK}\times$ $\times\exp\left[-m\sum_{j=2}^{\bar{n}}\tau_j\right]-$	0	0

b_{1i}	b_{2i}	b_{3i}	b_{4i}
$T^{(\bar{n})} - T_1 + \dfrac{q_0}{(h^2_{z(\bar{n})}a + m - aK)} - \dfrac{(T_1 - T_2)\,h_{z(\bar{n})}}{h^2_{z(\bar{n})} - K} - \left\{ \dfrac{H_{\bar{n}}(T_s - T_2)}{h^2_{z(\bar{n})}a - aK} \right\},\ i = \bar{n}$	$-\left\{ \dfrac{H_{\bar{n}}(T_s - T_2)}{h^2_{z(\bar{n})}a - aK} \right\},\ i = 1,$ $0,\ i \neq 1$	0	0
d_{1i}	d_{2i}	d_{3i}	d_{4i}
$-\dfrac{h_{z(\bar{n})}\sqrt{a}\,q_0}{\sqrt{\pi}\,(h^2_{z(\bar{n})}a + m - aK)} \times$ $\times \exp\left[-m \sum_{j=i+2}^{\bar{n}} \tau_j\right] \times$ $\times (1 - \exp[-m\tau_{i+1}]),\ i \neq \bar{n}$ $\dfrac{h_{z(\bar{n})}\sqrt{a}\,q_0}{\sqrt{\pi}\,(h^2_{z(\bar{n})}a + m - aK)},\ i = \bar{n}$	$\dfrac{h_{z(\bar{n})}\sqrt{a}\,q_0}{\sqrt{\pi}\,(h_{z(\bar{n})}a + m - aK)} \times$ $\times \exp\left[-m \sum_{j=2}^{\bar{n}} \tau_j\right],\ i = 1,$ $0,\ i \neq 1,$	0	0

g_{11}	g_{21}	g_{31}	g_{41}
$\frac{(h^2_{z(\bar n)}a - m + aK)q_0}{\sqrt{\pi}(m-aK)(h^2_{z(\bar n)}a + m - aK)} \times \exp\left[-m\sum_{j=i+2}^{\bar n}\tau_j\right] \times (1-\exp[-m\tau_{i+1}]),\ i \ne \bar n$; $\frac{2q_0 h^2_{z(\bar n)}a}{\sqrt{\pi}(m-aK)(h^2_{z(\bar n)}a + m - aK)},\ i = \bar n$	$\frac{q_0}{\sqrt{\pi}(m-aK)} \times \exp\left[-m\sum_{j=i+2}^{\bar n}\tau_j\right] \times (1-\exp[-m\tau_{i+1}]),\ i \ne \bar n$; $0,\ i = \bar n$	$\frac{(h^2_{z(\bar n)}a - m + aK)q_0}{\sqrt{\pi}(m-aK)(h^2_{z(\bar n)}a + m - aK)} \times \exp\left[-m\sum_{j=2}^{\bar n}\tau_j\right],\ i = 1$; $0,\ i \ne 1$	$\frac{q_0}{\sqrt{\pi}(m-aK)} \times \exp\left[-m\sum_{j=2}^{\bar n}\tau_j\right],\ i = 1$; $0,\ i \ne 1$
c_1	c_2	c_3	c_4
$\frac{(T^{ck}-T_2)(h_{z(\bar n)}+\sqrt{K})}{4(h_{z(\bar n)}-\sqrt{K})} + \left\{\frac{H_n(T_3-T_2)(h^2_{z(\bar n)}+\sqrt{K})}{aK(h_{z(\bar n)}-\sqrt{K})}\right\}$	$\frac{(T^{ck}-T_2)(h_{z(\bar n)}-\sqrt{K})}{4(h_{z(\bar n)}+\sqrt{K})} + \left\{\frac{H_n(T_3-T_2)\times(h_{z(\bar n)}-\sqrt{K})}{aK(h_{z(\bar n)}+\sqrt{K})}\right\}$	$\frac{(T_2-T_1)h_{z(\bar n)}}{2(h_{z(\bar n)}-\sqrt{K})} - \left\{\frac{H_n(T_3-T_2)h_{z(\bar n)}}{2(h_{z(\bar n)}-\sqrt{K})}\right\}$	$\frac{(T_2-T_1)h_{z(\bar n)}}{2(h_{z(\bar n)}+\sqrt{K})} - \left\{\frac{H_n(T_3-T_2)h_{z(\bar n)}}{2(h_{z(\bar n)}+\sqrt{K})}\right\}$

Notes: 1. For the spatial problem $K = K_{pr} + \mu_p^2/L^2 + x_r^2/D^2$, for the planar problem $K = \mu_p^2/L^2 + H_{\bar n}/a$. 2. The expressions in the braces are considered in the case of the planar problem, but are equal to 0 for the spatial problem.

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a\,\frac{\partial^2 T_0}{\partial z_{\bar{n}}^2} - H_{\bar{n}}(T_0 - T^{\text{cк}})$$

$$\left(\sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty,\ \tau > 0\right);$$

$$\frac{\partial T_s}{\partial \tau^{\bar{n}}} = a\,\frac{\partial^2 T_s}{\partial z_{\bar{n}}^2} - H_{\bar{n}}(T_s - T_s) + \frac{q_0}{c\gamma}\,e^{-mt},$$

$$\left(\sum_{j=s+1}^{\bar{n}} R_j < z_{\bar{n}} < \sum_{j=s}^{\bar{n}} R_j,\ \tau > 0;\ s = 1, 2, \ldots, \bar{n}\right);$$

(5-11)

The initial conditions

$$T_0(z_{\bar{n}}, 0) = T^{\text{cк}} + \Phi_0^{(\bar{n}-1)}(z_{\bar{n}}, \tau_{\bar{n}});$$

$$T_s(z_{\bar{n}}, 0) = E^{(\bar{n}-1)}\Big|_{\tau^{(\bar{n}-1)} = \tau_{\bar{n}}} (= E_s^{(\bar{n})}) + \Phi_s^{(\bar{n}-1)}(z_{\bar{n}}, \tau_{\bar{n}})$$

$$(s = 1, 2, \ldots, \bar{n} - 1);$$

$$T_{\bar{n}}(z_{\bar{n}}, 0) = E_{\bar{n}}^{(\bar{n})} = T^{(\bar{n})}.$$

(5-12)

The boundary conditions

$$\frac{\partial T_0(\infty, \tau^{(\bar{n})})}{\partial z_{\bar{n}}} = 0;\ T_0(\infty, \tau^{(\bar{n})}) \neq \infty;$$

$$\frac{\partial T_{\bar{n}}(0, \tau^{(\bar{n})})}{\partial z_{\bar{n}}} = -h_{\bar{n}}[T_1 - T_{\bar{n}}(0, \tau^{(\bar{n})})].$$

(5-13)

The conjugation conditions are the same as in the spatial problem. The solution of the problem is:

$$T_0(z_{\bar{n}}, \tau^{(\bar{n})}) = T^{\text{cк}} + \Phi_0^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})})$$

$$\left(\sum_{j=1}^{\bar{n}} R_j \leqslant z_{\bar{n}} < \infty, \tau > 0\right);$$

$$T_s(z_{\bar{n}}, \tau^{(\bar{n})}) = \Xi_s^{(\bar{n})}(\tau^{(\bar{n})}) + \Phi_c^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})})$$

$$\left(\sum_{j=s+1}^{\bar{n}} R_j \leqslant z_{\bar{n}} \leqslant \sum_{j=s}^{\bar{n}} R_j, \tau > 0;\ s = 1, 2, \ldots, \bar{n}\right), \tag{5-14}$$

where $E_s^{(\bar{n})}(\tau^{(\bar{n})})$ is a constant component of the temperature field (with respect to $z_{\bar{n}}$), equal to

$$\Xi_s^{(\bar{n})}(\tau^{(\bar{n})}) = T_s + \left(T^{(\bar{n})} - T_s + \frac{q_0}{c\gamma(m - H_{\bar{n}})}\right)\exp[-H_{\bar{n}}\tau^{(\bar{n})}] -$$

$$-\frac{q_0}{c\gamma(m - H_{\bar{n}})}\exp\left[-m\left(\sum_{j=s+1}^{\bar{n}} \tau_j + \tau^{(\bar{n})}\right)\right];$$

$\Phi_i^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})})$ is the variable component of the temperature field, equal to

$$\Phi_i^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) = F^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) + f_i^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) +$$

$$+ \exp[-H_{\bar{n}}\tau^{(\bar{n})}] \int_{R_{\bar{n}}}^{\infty} [\Phi^{(\bar{n}-1)}(z_{\bar{n}}, \xi, \tau_{\bar{n}})] \left\{\frac{1}{2\sqrt{\pi a\tau^{(\bar{n})}}} \times\right.$$

$$\times\left(\exp\left[-\frac{(z_{\bar{n}} - \xi)^2}{4a\tau^{(\bar{n})}}\right] + \exp\left[-\frac{(z_{\bar{n}} + \xi)^2}{4a\tau^{(\bar{n})}}\right]\right) -$$

$$- h_{\bar{n}}\exp[h_{\bar{n}}^2 a\tau^{(\bar{n})} + h_{\bar{n}}(z_{\bar{n}} + \xi)] \times$$

$$\left.\times \operatorname{erfc}\left[\frac{z_{\bar{n}} + \xi}{2\sqrt{a\tau^{(\bar{n})}}} + h_{\bar{n}}\sqrt{a\tau^{(\bar{n})}}\right]\right\} d\xi \quad (i = 0, c);$$

$$f_0^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) = \left(T^{(\bar{n})} - T_s + \frac{q_0}{c\gamma(m - H_{\bar{n}})}\right)\exp[-H_{\bar{n}}\tau^{(\bar{n})}] -$$

$$-\vartheta(z_{\bar{n}}, \tau^{(\bar{n})});$$

$$f_c^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) = \vartheta(z_{\bar{n}}, \tau^{(\bar{n})});$$

$$\theta(z_{\bar{n}}, \tau^{(\bar{n})}) = \frac{1}{4}(T^{\text{ск}} - T_3)\left(\exp\left[-\left(\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}\right)\sqrt{\frac{H_{\bar{n}}}{a}}\right] \times \right.$$

$$\times \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}} - \sqrt{H_{\bar{n}}\tau^{(\bar{n})}}\right] +$$

$$\left. + \exp\left[\left(\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}\right)\sqrt{\frac{H_{\bar{n}}}{a}}\right] \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}} R_j - z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}} + \sqrt{H_{\bar{n}}\tau^{(\bar{n})}}\right]\right);$$

$$\dot{F}^{(\bar{n})}(z_{\bar{n}}, \tau^{(\bar{n})}) = W_{II}(z_{\bar{n}}, \tau^{(\bar{n})}). \tag{5-15}$$

The values of coefficients $a_{\rho i}$ and $b_{\rho i}$ are given in Table 5-2. The remaining functions are the same as for the spatial problem, but we should replace K_{pr} by $H_{\bar{n}}/a$.

As in the case of the spatial and planar problems, calculation of the integral in the expression (5-15) is based on approximation of the function $\Phi^{(\bar{n}-1)}$ by the polynomial

$$\Phi^{(n-1)} = \sum_{i=0}^{\nu} \alpha_i z^i.$$

The coefficients of the polynomial α_i are determined as a result of solution of a system of equations produced for the values of the function $\Phi^{(\bar{n}-1)}$ at $(\nu + 1)$ equally spaced points. The recurrent formulas for calculation of α_i are presented below.

The analytic solutions to the problem of the two-dimensional and one-dimensional temperature field of a concrete mass constructed of individual blocks which we produced have been used as algorithms for programming the corresponding calculations by computer.

In addition to the general information on the concrete mass (heat-physical characteristics, heat liberation intensity function), the process of calculation of a one-dimensional temperature field includes data on the next block, including the height of the block, the initial temperature of the

TABLE 5-2. VALUES OF COEFFICIENTS $a_{\rho i}$ AND $b_{\rho i}$ FOR THE ONE-DIMENSIONAL PROBLEM

a_{1i}	a_{2i}	a_{3i}	a_{4i}
$\frac{1}{2}\left\{E_{i+1}^{(\bar{n})} - E_i^{(\bar{n})} + \frac{q_0}{m - H_{\bar{n}}} \times \exp\left[-m \sum_{j=i+2}^{\bar{n}} \tau_j\right] \times (1 - \exp[-m\tau_{i+1}])\right.$, $i \neq \bar{n}$; $T_3 - T^{(\bar{n})} + \frac{q_0}{m - H_{\bar{n}}}$, $i = \bar{n}$	$-a_{1i}$, $i \neq \bar{n}$; 0, $i = \bar{n}$	$\frac{1}{2}\left(E_i^{(\bar{n})} - T_3 + \frac{q_0}{m - H_{\bar{n}}} \times \exp\left[-m \sum_{j=2}^{\bar{n}} \tau_j\right]\right)$, $i = 1$; 0, $i \neq 1$	$-a_3$

b_{1i}	b_{2i}	b_{3i}	b_{4i}
$E_i^{(\bar{n})} - E_{i+1}^{(\bar{n})} - \frac{q_0}{h_{\bar{n}}^2 a + m - H_{\bar{n}}} \times \exp\left[-m \sum_{j=i+2}^{\bar{n}} \tau_j\right] \times (1 - \exp[-m\tau_{i+1}])$, $i \neq \bar{n}$; $T^{(\bar{n})} - T_1 + \frac{q_0}{h_{\bar{n}}^2 a + m - H_{\bar{n}}} + \frac{H_{\bar{n}}(T_3 - T_1)}{h_{\bar{n}}^2 a - H_{\bar{n}}}$, $i = \bar{n}$	$T^{\text{ск}} - E_i^{(\bar{n})} - \frac{q_0}{h^2 a + m - H_{\bar{n}}} \times \exp\left[-m \sum_{j=2}^{\bar{n}} \tau_j\right] + \frac{H_{\bar{n}}(T^{\text{ск}} - T_3)}{h_{\bar{n}}^2 a - H_{\bar{n}}}$, $i = 1$; 0, $i \neq 1$	0	0

concrete, the ambient temperature, parameter H, the time interval between coverage of the block by the next block and the assigned number of calculation points with respect to time. The program is used to calculate the values of temperature at 6 points through the height of both the concrete block and its reflection in the base.

Basically, the logic system of the program for calculation of a two-dimensional temperature field is similar to the plan for the program for calculation of a one-dimensional field. One peculiarity is the use of peripheral

memory, as well as the approach used for consideration of the effect of the base on the thermal mode of the mass. This approach is as follows.

The integral of function $\phi^{(\bar{n}-1)}$ is divided into two

$$\int_{R_{\bar{n}}}^{\sum_{j=1}^{\bar{n}}(R_j+R_{-j})} [\Phi^{(\bar{n}-1)}]\{\ \}\,d\xi = \int_{R_{\bar{n}}}^{\sum_{j=1}^{\bar{n}} R_j} + \int_{\sum_{j=1}^{\bar{n}} R_j}^{\sum_{j=1}^{\bar{n}}(R_j+R_{-j})}.$$

The first integral in the right portion of this expression is calculated for each value of x_k, the second integral, related to the base, is calculated only for points on the axis of the mass ($x_k = 0$), and this value is assigned to the remaining x_k. Thus, we have artificially created additional heat insulation of the side surface of the mass in the area of the base.

Algorithms for Calculation of Certain Integrals

1. The function erfc x

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}dx. \qquad (5\text{-}16)$$

Here

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.$$

The numerical values of erfc x where x < 2 can be conveniently determined from the following formula [54, 69]

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}. \qquad (5\text{-}17)$$

In order to calculate erfc x where x $\geq$ 2, we recommend the approximate formula

$$\operatorname{erfc} x = \frac{2e^{-x^2}}{\sqrt{\pi}} \frac{\sum_{i=0}^{5} a_i x^{2i+1}}{\sum_{i=0}^{6} b_i x^{2i}}, \tag{5-18}$$

where

$$\begin{aligned} a_0 &= 35\,685, & b_0 &= 10\,395, \\ a_1 &= 83\,370, & b_1 &= 124\,740, \\ a_2 &= 50\,232, & b_2 &= 207\,900, \\ a_3 &= 11\,376, & b_3 &= 110\,880, \\ a_4 &= 1\,040, & b_4 &= 23\,760, \\ a_5 &= 32, & b_5 &= 2\,112, \\ & & b_6 &= 64. \end{aligned}$$

Algorithm (5-18) was produced from the following considerations.

We know that

$$\mathrm{i}^n \operatorname{erfc} x \to 0 \text{ при } n \to \infty \tag{5-18'}$$

and

$$2n\, \mathrm{i}^n \operatorname{erfc} x = \mathrm{i}^{n-2} \operatorname{erfc} x - 2x\, \mathrm{i}^{n-1} \operatorname{erfc} x,$$

where

$$\mathrm{i}^n \operatorname{erfc} x = \int_x^\infty \mathrm{i}^{n-1} \operatorname{erfc} x\, dx;$$

$$\mathrm{i}^0 \operatorname{erfc} x = \operatorname{erfc} x;$$

$$\mathrm{i}^{-1} \operatorname{erfc} x = \frac{d}{dx} \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

Consequently, with sufficiently great n, condition (5-18') can be looked upon as the equation for erfc x. In producing formula (5-18), it was assumed that

$$i^{12}\operatorname{erfc} x/_{x=2}=0.$$

(Estimates give $i^{12}\operatorname{erfc} x/_{x=2} < 10^{-11}$.)

Table 5-3 presents the results of comparison of the values of erfc x calculated by algorithm (5-18) on a computer with the tabular data for the probabilistic functions erfc x [120].

TABLE 5-3. VALUES OF THE FUNCTION erfc x
[Tabular Data of [120] and Results of Calculation Using Algorithm (5-18)]

x	erfc x	
	1 Табличные данные [120] (округленные)	2 Значения, вычисленные по алгоритму (5-18)
2,0	$0{,}0^{2}46777360$	$0{,}0^{2}46777326$
2,2	$0{,}0^{2}18628463$	$0{,}0^{2}18628461$
2,4	$0{,}0^{3}68851390$	$0{,}0^{3}68851388$
2,6	$0{,}0^{3}23603442$	$0{,}0^{3}23603442$
3,0	$0{,}0^{4}22090497$	$0{,}0^{4}22090497$
3,6	$0{,}0^{6}35586299$	$0{,}0^{6}35586301$
4,0	$0{,}0^{7}15417257$	$0{,}0^{7}15417259$

Key: 1, Tabular Data [120] (Rounded); 2, Values Calculated by Algorithm (5-18)

As we can see from Table 5-3, for values of the argument $x \geq 2$, algorithm (5-18) calculates function erfc x with an error of 10^{-7}-10^{-8}.

2. The integral

$$I_{(2k+1)/2}\left(m-aK_{pr.}-\left(\frac{N}{2\sqrt{a}}\right)^{2},\ \tau^{(\bar{n})}\right)=$$
$$=\left(\frac{N}{2\sqrt{a}}\right)^{k}\int_{0}^{\tau^{(\bar{n})}}\vartheta^{-\frac{2k+1}{2}}\exp\left[(m-aK_{pr})(\vartheta-\tau^{(\bar{n})})-\frac{N^{2}}{4a\vartheta}\right]d\vartheta.$$

The factor $(m - aK_{pr})$ can take on both positive and negative values.

Where $(m - aK_{pr}) < 0$, we are concerned with the calculation of an integral of the type

$$\int_0^\tau \vartheta^{-\frac{2k+1}{2}} \exp\left[-\lambda\vartheta - \frac{N^2}{4a\vartheta}\right] d\vartheta \quad (\lambda > 0;\ k = 0, 1).$$

Let us write the identity

$$\vartheta^{-\frac{1}{2}} \exp\left[-\lambda\vartheta - \frac{N^2}{4a\vartheta}\right] = \frac{1}{\sqrt{\lambda}}\left\{\exp\left[N\sqrt{\frac{\lambda}{a}}\right]\exp\left[-\left(\sqrt{\lambda\vartheta} + \frac{N}{2\sqrt{a\vartheta}}\right)^2\right]\left(\sqrt{\lambda\vartheta} + \frac{N}{2\sqrt{a\vartheta}}\right)' - \exp\left[-N\sqrt{\frac{\lambda}{a}}\right]\exp\left[-\left(\sqrt{\lambda\vartheta} - \frac{N}{2\sqrt{a\vartheta}}\right)^2\right]\left(\frac{N}{2\sqrt{a\vartheta}} - \sqrt{\lambda\vartheta}\right)'\right\} = -\frac{\sqrt{\pi}}{2\sqrt{\lambda}}\left\{\exp\left[N\sqrt{\frac{\lambda}{a}}\right] \times \left(\operatorname{erfc}\left[\frac{N}{2\sqrt{a\vartheta}} + \sqrt{\lambda\vartheta}\right]\right)' - \exp\left[-N\sqrt{\frac{\lambda}{a}}\right] \times \left(\operatorname{erfc}\left[\frac{N}{2\sqrt{a\vartheta}} - \sqrt{\lambda\vartheta}\right]\right)'\right\}.$$

Taking the integral from 0 to τ from both portions of the identity, we find

$$\int_0^\tau \vartheta^{-\frac{1}{2}} \exp\left[-\lambda\vartheta - \frac{N^2}{4a\vartheta}\right] d\vartheta = \frac{\sqrt{\pi}}{2\sqrt{\lambda}} \exp\left[-N\sqrt{\frac{\lambda}{a}}\right] \operatorname{erfc}\left[\frac{N}{2\sqrt{a\tau}} - \sqrt{\lambda\tau}\right] - \frac{\sqrt{\pi}}{2\sqrt{\lambda}} \exp\left[N\sqrt{\frac{\lambda}{a}}\right] \operatorname{erfc}\left[\frac{N}{2\sqrt{a\tau}} + \sqrt{\lambda\tau}\right]. \tag{5-19}$$

Similarly, we can show that

$$\int_0^\tau \vartheta^{-\frac{3}{2}} \exp\left[-\lambda\vartheta - \frac{N^2}{4a\vartheta}\right] d\vartheta = \frac{\sqrt{\pi a}}{N} \exp\left[-N\sqrt{\frac{\lambda}{a}}\right] \operatorname{erfc}\left[\frac{N}{2\sqrt{a\tau}} - \sqrt{\lambda\tau}\right] + \frac{\sqrt{\pi a}}{N} \exp\left[N\sqrt{\frac{\lambda}{a}}\right] \operatorname{erfc}\left[\frac{N}{2\sqrt{a\tau}} + \sqrt{\lambda\tau}\right]. \tag{5-20}$$

Where $(m - aK_{pr}) > 0$, we must calculate integrals of the type

$$\int_0^{\tau} \vartheta^{-\frac{2k+1}{2}} \exp\left[\lambda\vartheta - \frac{N^2}{4a\vartheta}\right] d\vartheta \quad (\lambda > 0;\ k = 0, 1).$$

We suggest below a representation of integrals $I_{(2k+1)/2}(b, -c^2, \tau)$ where $b > 0$ in the form of infinite series, which essentially simplifies calculations.

As we know

$$e^{b\vartheta} = \sum_{n=0}^{\infty} \frac{b^n \vartheta^n}{n!}.$$

From this

$$I_{3/2}(b, -c^2, \tau) = c \int_0^{\tau} \vartheta^{-\frac{3}{2}} \exp\left[b(\vartheta - \tau) - \frac{c^2}{\vartheta}\right] d\vartheta =$$
$$= c e^{-b\tau}\left[J_1 + \frac{b}{1!} J_2 + \frac{b^2}{2!} J_3 + \ldots\right],$$

where

$$J_1 = \int_0^{\tau} \vartheta^{-3/2} e^{-\frac{c^2}{\vartheta}} d\vartheta;$$
$$J_2 = \int_0^{\tau} \vartheta^{-1/2} e^{-\frac{c^2}{\vartheta}} d\vartheta;$$
$$J_3 = \int_0^{\tau} \vartheta^{1/2} e^{-\frac{c^2}{\vartheta}} d\vartheta;$$
$$\cdots\cdots\cdots .$$

Integral $I_{1/2}(b, -c^2, \tau)$ is equal to

$$I_{1/2}(b, -c^2, \tau) = \int_0^{\tau} \vartheta^{-\frac{1}{2}} \exp\left[b(\vartheta - \tau) - \frac{c^2}{\vartheta}\right] d\vartheta =$$
$$= e^{-b\tau}\left[J_2 + \frac{b}{1!} J_3 + \frac{b^2}{2!} J_4 + \ldots\right].$$

It is not difficult to see that integrals J are incomplete gamma functions $\Gamma(\alpha, x)$ [25], namely

$$J_1 = \int_0^{\tau} \vartheta^{-\frac{3}{2}} e^{-\frac{c^2}{\vartheta}} d\vartheta = \frac{1}{c} \int_{c^2/\tau}^{\infty} t^{\frac{1}{2}-1} e^{-t} dt =$$
$$= \frac{1}{c} \Gamma\left(\frac{1}{2}, \frac{c^2}{\tau}\right);$$

$$J_2 = \int_0^{\tau} \vartheta^{-\frac{1}{2}} e^{-\frac{c^2}{\vartheta}} d\vartheta = c\Gamma\left(-\frac{1}{2}, \frac{c^2}{\tau}\right);$$

$$J_3 = \int_0^{\tau} \vartheta^{\frac{1}{2}} e^{-\frac{c^2}{\vartheta}} d\vartheta = c^3\Gamma\left(-\frac{3}{2}, \frac{c^2}{\tau}\right);$$

$$\cdots\cdots\cdots\cdots\cdots$$

However

$$\Gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x})$$

and

$$\Gamma(\alpha+1, x) = \alpha\Gamma(\alpha, x) + x^{\alpha} e^{-x}.$$

Using these relationships, we can suggest for calculation of the integrals $I_{(2k+1)/2}(b, -c^2, \tau)$ an algorithm of simple form

$$I_{(2k+1)/2}(b, -c^2, \tau) = \left[G_{-1} + \sum_{n=0}^{\infty} G_n\right] \quad (k = 0, 1, 2\ldots), \tag{5-21}$$

where

$$G_n = \frac{2}{(n+1)(2n+3-2k)}[D_n - bc^2 G_{n-1}];$$

$$G_{-1} = c^{-(2k-1)}\Gamma\left(\frac{2k-1}{2}, \frac{c^2}{\tau}\right);$$

$$D_0 = \tau^{-\frac{2k-1}{2}} e^{\left(-q - \frac{c^2}{\tau}\right)q};$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$D_n = D_{n-1}\frac{q}{n};$$

$$q = b\tau,\ \Gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x}),\ \Gamma(\alpha+1) = \alpha\Gamma(\alpha, x) + x^{\alpha}e^{-x}.$$

Analysis has shown that with the same values of b and c encountered in calculation of temperature fields in concrete masses, the series in expression (5-21) converges rather rapidly.

Based on algorithm (5-21), a standard program has been composed for computer calculation of integrals of this type. The values of integrals $I_{1/2}(b, -c^2, \tau)$ and $I_{3/2}(b, -c^2, \tau)$ calculated by this program are presented in [104].

Algorithm (5-21) is suitable for determination of numerical values of the integrals $I_{(2k-1)/2}(b, -c^2, \tau)$ with any positive integer values of k. In addition to this, it is simple to produce recurrent relationships which can be used to calculate the integrals $I_{(2k+1)/2}$ where $k > 1$, using the tabular values of $I_{1/2}$ and $I_{3/2}$.

Let us find the derivative $\frac{d}{d\tau}\frac{e^{b\tau - c^2/\tau}}{\tau^{1/2}} = b\frac{e^{b\tau - c^2/\tau}}{\tau^{1/2}} + c^2\frac{e^{b\tau - c^2/\tau}}{\tau^{5/2}} - \frac{1}{2}\frac{e^{b\tau - c^2/\tau}}{\tau^{3/2}}.$

Let us integrate the right and left portions of this equation within limits of 0 to τ and multiply the result by $e^{-b\tau}$. We produce:

$$cI_{5/2} = c\tau^{-1}e^{-c^2/\tau} - bcI_{1/2} + \frac{1}{2}I_{3/2}.$$

We can similarly show that there are recurrent formulas of a more general form

$$cI_{(2k+1)/2} = c^{k-1}\tau^{-(2k-3)/2}e^{-c^2/\tau} - bcI_{(2k-3)/2} + \frac{(2k-3)}{2}I_{(2k-1)/2}$$
$$(k = 2,\ 3\ \ldots).$$

3. The integral

$$\int\limits_{R_{\bar{n}}}^{\infty}[\Phi^{(\bar{n}-1)}(z_n,\xi,\tau_{\bar{n}})]\left\{\frac{1}{2\sqrt{\pi a\tau^{(\bar{n})}}}\left(\exp\left[-\frac{(z_{\bar{n}}-\xi)^2}{4a\tau^{(\bar{n})}}\right]+\right.\right.$$
$$\left.+\exp\left[-\frac{(z_{\bar{n}}+\xi)^2}{4a\tau^{(\bar{n})}}\right]\right)-h_{\bar{n}}\exp[h_{\bar{n}}^2a\tau^{(\bar{n})}+h_{\bar{n}}(z_{\bar{n}}+\xi)]\times$$
$$\left.\times\operatorname{erfc}\left[h_{\bar{n}}\sqrt{a\tau^{(\bar{n})}}+\frac{z_{\bar{n}}+\xi}{2\sqrt{a\tau^{(\bar{n})}}}\right]\right\}d\xi. \tag{5-22}$$

Calculation of integral (5-22) is based on approximation of the function $\Phi^{(\bar{n}-1)}$ within the limits of each block, real and "imaginary," by the polynomial

$$\Phi^{(\bar{n}-1)}=\sum_{i=0}^{v}\alpha_i z^i.$$

Here, partial integrals appear, which are equal to

$$\int\limits_a^b\Phi^{(\bar{n}-1)}\left\{\frac{1}{2\sqrt{\pi a\tau}}\left(\exp\left[-\frac{(z-\xi)^2}{4a\tau}\right]+\exp\left[-\frac{(z+\xi)^2}{4a\tau}\right]\right)-\right.$$
$$\left.-h\exp[h^2a\tau+h(z+\xi)]\operatorname{erfc}\left[\frac{z+\xi}{2\sqrt{a\tau}}+h\sqrt{a\tau}\right]\right\}d\xi=\sum_{i=0}^{v}\alpha_i I_i, \tag{5-23}$$

where

$$I_0=J_0^{(1)}-J_0^{(2)}+J_0^{(3)};$$
$$I_1=J_1^{(1)}-J_1^{(2)}+J_1^{(3)};$$
$$\cdots\cdots\cdots\cdots$$

$$I_\nu = J_\nu^{(1)} - J_\nu^{(2)} + J_\nu^{(3)};$$

$$J_0^{(1)} = \frac{1}{2}\left(\operatorname{erfc}\left[\frac{a-z}{2\sqrt{a\tau}}\right] - \operatorname{erfc}\left[\frac{b-z}{2\sqrt{a\tau}}\right]\right);$$

$$J_0^{(2)} = \frac{1}{2}\left(\operatorname{erfc}\left[\frac{a+z}{2\sqrt{a\tau}}\right] - \operatorname{erfc}\left[\frac{b+z}{2\sqrt{a\tau}}\right]\right);$$

$$J_0^{(3)} = \exp[h^2 a\tau + h(a+z)]\operatorname{erfc}\left[\frac{a+z}{2\sqrt{a\tau}} + h\sqrt{a\tau}\right] -$$

$$- \exp[h^2 a\tau + h(b+z)]\operatorname{erfc}\left[\frac{b+z}{2\sqrt{a\tau}} + h\sqrt{a\tau}\right];$$

$$J^{(1)}_{s\,(s>0)} = zJ^{(1)}_{s-1} + 2(s-1)a\tau J^{(1)}_{s-2} + \sqrt{\frac{a\tau}{\pi}}\left(a^{s-1}\exp\left[-\frac{(a-z)^2}{4a\tau}\right] -\right.$$

$$\left. - b^{s-1}\exp\left[-\frac{(b-z)^2}{4a\tau}\right]\right);$$

$$J^{(2)}_{s\,(s>0)} = -zJ^{(2)}_{s-1} + 2(s-1)a\tau J^{(2)}_{s-2} +$$

$$+ \sqrt{\frac{a\tau}{\pi}}\left(a^{s-1}\exp\left[-\frac{(a+z)^2}{4a\tau}\right] - b^{s-1}\exp\left[-\frac{(b+z)^2}{4a\tau}\right]\right);$$

$$J^{(3)}_{s\,(s>0)} = \frac{2s}{h}J^{(2)}_{s-1} - \frac{2s(s-1)}{h^2}J^{(2)}_{s-2} + \frac{2s(s-1)(s-2)}{h^4}J^{(2)}_{s-3} -$$

$$- \frac{2s(s-1)(s-2)(s-3)}{h^4}J^{(2)}_{s-4} + \left(a^s - \frac{s}{h}a^{s-1} + \frac{s(s-1)}{h^2}a^{s-2} -\right.$$

$$- \frac{s(s-1)(s-2)}{h^3}a^{s-3} + \frac{s(s-1)\dots(s-3)}{h^4}a^{s-4} -$$

$$- \frac{s(s-1)\dots(s-4)}{h^5}a^{s-5} + \dots + \frac{s(s-1)\dots[s-(\nu-2)]}{h^{\nu-1}}a^{s-(\nu-1)} -$$

$$\left. - \frac{s(s-1)\dots[s-(\nu-1)]}{h^\nu}\right)\exp[h^2 a\tau + h(a+z)]\operatorname{erfc}\left[\frac{a+z}{2\sqrt{a\tau}} +\right.$$

$$\left. + h\sqrt{a\tau}\right] - \left(b^s - \frac{s}{h}b^{s-1} + \frac{s(s-1)}{h^2}b^{s-2} - \frac{s(s-1)(s-2)}{h^3}b^{s-3} +\right.$$

$$+ \dots + \frac{s(s-1)\dots[s-(\nu-2)]}{h^{\nu-1}}b^{s-(\nu-1)} -$$

$$\left. - \frac{s(s-1)\dots[s-(\nu-1)]}{h^\nu}\right)\exp[h^2 a\tau + h(b+z)] \times$$

$$\times \operatorname{erfc}\left[\frac{b+z}{2\sqrt{a\tau}} + h\sqrt{a\tau}\right].$$

5-3. Consideration of Supplementary Factors

In this section, the basic solutions produced above are extended to more complex cases of calculation of temperature fields of concrete masses, encountered in engineering practice.

Consideration of Variation of Ambient Temperature with Time

In the solutions of § 5-2, it was assumed that during the time interval between coverage of a block with another block, the ambient temperature was constant. This condition is usually sufficient for description of the thermal state of the mass: during the relatively brief intervals of coverage called for in the plan, the ambient temperature changes but slightly, in regions with severe climate concrete is poured beneath a tent, where a constant temperature is artificially maintained.

However, under actual construction conditions due to disruptions of the concrete pouring schedule, the coverage time is sometimes so greatly extended that we cannot ignore the change in ambient temperature.

The computer calculation program based on the solutions of § 5-2 called for approximation of the dependence of ambient temperature on time as a piecewise-constant function. It was reduced to introduction at moments of sudden change in the ambient temperature of "0" blocks, i.e., blocks, the height of which is equal to 0. We present below a precise analytic solution of the problem.

1. Spatial temperature field. Suppose the mass consists of $\bar{n}$ blocks and the ambient temperature is $T_{cp}^{(1)} = T_1 + \phi_{\bar{n}}(\tau^{(\bar{n})})$ on the horizontal surface and $T_{cp}^{(2)} = T_2 + \psi_{\bar{n}}(\tau^{(\bar{n})})$ on the vertical surfaces.

Let us represent the general solution to the problem as

$$T = (T)_1 + (T)_2,$$

where $(T)_1$ is the solution produced in § 5-2, considering the heat liberation in the concrete, initial temperature of the mass and constant components of the ambient temperature T_1 and T_2; $(T)_2$ is the solution of the problem for a mass without heat liberation, with zero initial temperature and variable components of the ambient temperature

$$\varphi_{\bar{n}}(\tau^{(\bar{n})}) \text{ и } \psi_{\bar{n}}(\tau^{(\bar{n})}).$$

Thus, in order to determine $(T)_2$, we must solve the following problem:

$$\frac{\partial T}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T}{\partial z_{\bar{n}}^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$(0 < z_{\bar{n}} < \infty, \ -L < x < L, \ -D < y < D, \ \tau^{(\bar{n})} > 0);$$

$$T(z_{\bar{n}}, x, y, 0) = 0$$

$$(0 \le z_{\bar{n}} < \infty, \ -L \le x \le L, \ -D \le y \le D);$$

$$\frac{\partial T(0, x, y, \tau^{(\bar{n})})}{\partial z_{\bar{n}}^2} = -h_{z(\bar{n})}[\varphi_{\bar{n}}(\tau^{(\bar{n})}) - T(0, x, y, \tau^{(\bar{n})})]; \tag{5-24}$$

$$\frac{\partial T(\infty, x, y, \tau^{(\bar{n})})}{\partial z_{\bar{n}}^2} = 0;$$

$$\frac{\partial T(z_{\bar{n}}, L, y, \tau^{(\bar{n})})}{\partial x} = h_{x(\bar{n})}[\Psi_{\bar{n}}(z_{\bar{n}}, \tau^{(\bar{n})}) - T(z_{\bar{n}}, L, y, \tau^{(\bar{n})})];$$

$$\frac{\partial T(z_{\bar{n}}, 0, y, \tau^{(\bar{n})})}{\partial x} = 0;$$

$$\frac{\partial T(z_{\bar{n}}, x, D, \tau^{(\bar{n})})}{\partial y} = h_{y(\bar{n})}[\Psi_{\bar{n}}(z_{\bar{n}}, \tau^{(\bar{n})}) - T(z_{\bar{n}}, x, D, \tau^{(\bar{n})})];$$

$$\frac{\partial T(z_{\bar{n}}, x, 0, \tau^{(\bar{n})})}{\partial y} = 0,$$

where

$$\Psi_{\bar{n}}(z_{\bar{n}}, \tau^{(\bar{n})}) = \begin{cases} \varphi_{\bar{n}}(\tau^{(\bar{n})}) \text{ where } 0 < z_{\bar{n}} < \sum_{j=1}^{\bar{n}} R_j \text{ (area of concrete mass);} \\ 0 \quad \text{where } \sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty \text{(are of base).} \end{cases}$$

The result

$$T = \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} aK_{pr}A_pB_r \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \exp[-aK_{pr}\tau^{(\bar{n})}] \times$$
$$\times \int_0^{\tau^{(\bar{n})}} \varphi_{\bar{n}}(t) e^{aK_{pr}t}\, \Pi(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt +$$
$$+ ah_{z(\bar{n})} \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_pB_r \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \exp[-aK_{pr}\tau^{(\bar{n})}] \times$$
$$\times \int_0^{\tau^{(\bar{n})}} \varphi_{\bar{n}}(t)\, e^{aK_{pr}t}\, E(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt, \tag{5-25}$$

where

$$K_{pr} = \frac{\mu_p^2}{L^2} + \frac{\varkappa_r^2}{D^2};$$

$$\Pi(z_{\bar n}, \tau^{(\bar n)} - t) = \operatorname{erf}\left[\frac{z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}}\right] + \frac{1}{2}\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar n} R_j + z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}}\right] -$$

$$-\frac{1}{2}\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar n} R_j - z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}}\right] + \exp[h_{z(\bar n)}^2 a(\tau^{(\bar n)} - t) +$$

$$+ h_{z(\bar n)} z_{\bar n}] \operatorname{erfc}\left[\frac{z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}} + h_{z(\bar n)}\sqrt{a(\tau^{(\bar n)} - t)}\right] -$$

$$- \exp\left[h_{z(\bar n)}^2 a(\tau^{(\bar n)} - t) + h_{z(\bar n)}\left(\sum_{j=1}^{\bar n} R_j + z_{\bar n}\right)\right] \times$$

$$\times \operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar n} R_j + z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}} + h_{z(\bar n)}\sqrt{a(\tau^{(\bar n)} - t)}\right];$$

$$E(z_{\bar n}, \tau^{(\bar n)} - t) = \frac{1}{\sqrt{\pi a(\tau^{(\bar n)} - t)}} \exp\left[-\frac{z_{\bar n}^2}{4a(\tau^{(\bar n)} - t)}\right] -$$

$$- h_{z(\bar n)} \exp[h_{z(\bar n)}^2 a(\tau^{(\bar n)} - t) + h_{z(\bar n)} z_{\bar n}] \times$$

$$\times \operatorname{erfc}\left[\frac{z_{\bar n}}{2\sqrt{a(\tau^{(\bar n)} - t)}} + h_{z(\bar n)}\sqrt{a(\tau^{(\bar n)} - t)}\right].$$

2. Two-dimensional (planar) temperature field. The differential equation

$$\frac{\partial T}{\partial \tau^{(\bar n)}} = a\left(\frac{\partial^2 T}{\partial z_{\bar n}^2} + \frac{\partial^2 T}{\partial x^2}\right) - H_{\bar n}[T - \Omega_{\bar n}(z_{\bar n}, \tau^{(\bar n)})]$$

$$(0 < z_{\bar n} < \infty, \ -L < x < L, \ \tau^{(\bar n)} > 0). \qquad (5\text{-}26)$$

The initial and boundary conditions are the same as in (5-24), but we must keep in mind that the problem is two-dimensional.

Function $\Omega_{\bar{n}}$ is equal to:

$$\Omega_{\bar{n}}(z_{\bar{n}}, \tau^{(\bar{n})}) = \begin{cases} Z_{\bar{n}}(\tau^{(\bar{n})}) & \text{where} \quad 0 < z_{\bar{n}} < \sum_{j=1}^{\bar{n}} R_j; \\ 0 & \text{where} \quad \sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty. \end{cases}$$

The solution

$$T = \sum_{p=1}^{\infty} aA_p \cos\mu_p \frac{x}{L} e^{-aK_{pH}\tau^{(\bar{n})}} \int_0^{\tau^{(\bar{n})}} \left[\frac{\mu_p^2}{L^2} \varphi_{\bar{n}}(t) + \right.$$

$$\left. + \frac{H_{\bar{n}}}{a} Z_{\bar{n}}(t) \right] e^{aK_{pH}t}\, \Omega(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt +$$

$$+ ah_{z(\bar{n})} \sum_{p=1}^{\infty} A_p \cos\mu_p \frac{x}{L} e^{-aK_{pH}\tau^{(\bar{n})}} \times$$

$$\times \int_0^{\tau^{(\bar{n})}} \varphi_{\bar{n}}(t)\, e^{aK_{pH}t}\, E(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt, \qquad (5\text{-}27)$$

where

$$K_{pH} = \frac{\mu_p^2}{L^2} + \frac{H_{\bar{n}}}{a}.$$

3. One-dimensional temperature field

$$\frac{\partial T}{\partial \tau^{(\bar{n})}} = a \frac{\partial^2 T}{\partial z_{\bar{n}}^2} - H_{\bar{n}} [T - \Omega_{\bar{n}}(z_{\bar{n}}, \tau^{(\bar{n})})];$$

$$T(z_{\bar{n}}, 0) = 0;$$

$$\frac{\partial T(0, \tau^{(\bar{n})})}{\partial z_{\bar{n}}} = -h_{z(\bar{n})} [\varphi_{\bar{n}}(\tau^{(\bar{n})}) - T(0, \tau^{(\bar{n})})];$$

$$\frac{\partial T(\infty, \tau^{(\bar{n})})}{\partial z_{\bar{n}}} = 0. \qquad (5\text{-}28)$$

The solution

$$T = H_{\bar{n}} e^{-H_{\bar{n}} \tau^{(\bar{n})}} \int_0^{\tau^{(\bar{n})}} Z_{\bar{n}}(t) e^{H_{\bar{n}} t} \text{П}(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt +$$

$$+ a h_{z(\bar{n})} e^{-H_{\bar{n}} \tau^{(\bar{n})}} \int_0^{\tau^{(\bar{n})}} \varphi_{\bar{n}}(t) e^{H_{\bar{n}} t} E(z_{\bar{n}}, \tau^{(\bar{n})} - t)\, dt. \qquad (5\text{-}29)$$

The primary difficulty in calculating temperature fields using the formulas presented (5-25), (5-27) and (5-29) is the calculation of the integrals containing the time-variable ambient temperature. Considering the relatively slight time intervals between coverage of blocks with blocks, we can suggest approximation of the ambient temperature with an exponential function. The calculations in this case are greatly simplified, being reduced to summation of integrals such as

$$\sum_{s=1}^{k-1} \int_{\tau_s}^{\tau_{s+1}} e^{-\lambda(\tau - t)} E(z, \tau - t)\, dt$$

and

$$\sum_{s=1}^{k-1} \int_{\tau_s}^{\tau_{s+1}} e^{-\lambda(\tau - t)} \text{П}(z, \tau - t)\, dt.$$

The partial integrals of this sum are:

$$\int_{\tau_s}^{\tau_{s+1}} = \int_0^{\tau_{s+1}} - \int_0^{\tau_s},$$

$$\int_0^{\tau_s} e^{-\lambda(\tau - t)} E(z, \tau - t)\, dt = \frac{(h\sqrt{\lambda a} - \lambda)}{2\sqrt{\lambda a}\,(h^2 a - \lambda)} \times$$

$$\times \sum_{p=1}^{2} \left\{ (-1)^{p-1} e^{(-1)^p \sqrt{\frac{\lambda}{a}}} \operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau_s}} + (-1)^p \sqrt{\lambda\tau_s}\right]\right\} +$$

$$+ \frac{h}{(h^2 a - \lambda)} \exp\left[(h^2 a - \lambda)\tau_s + hz\right] \operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau_s}} + h\sqrt{a\tau_s}\right];$$

$$\int_0^{\tau_s} e^{-\lambda(\tau - t)} \text{П}(z, \tau - t)\, dt = -\frac{1}{\lambda} e^{-\lambda\tau_s} \operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau_s}}\right] +$$

$$
+\frac{1}{2\lambda}\sum_{\rho=1}^{2}(-1)^{\rho}\left\{e^{-\lambda\tau_s}\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}}R_j+(-1)^{\rho-1}z}{2\sqrt{a\tau_s}}\right]+\right.
$$

$$
+\frac{[h^2a+(-1)^{\rho-1}h\sqrt{\lambda a}-(2\lambda)^{\rho-1}]}{(h^2a-\lambda)}e^{(-1)^{\rho}z\sqrt{\frac{\lambda}{a}}}\times
$$

$$
\times\operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau_s}}+(-1)^{\rho}\sqrt{\lambda\tau_s}\right]-
$$

$$
-\frac{1}{2}e^{(-1)^{\rho}\sqrt{\frac{\lambda}{a}}}\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}}R_j+z}{2\sqrt{a\tau_s}}+(-1)^{\rho}\sqrt{\lambda\tau_s}\right]+
$$

$$
\left.+\frac{1}{2}e^{(-1)^{\rho}\sqrt{\frac{\lambda}{a}}}\operatorname{erfc}\left[\frac{\sum_{j=1}^{\bar{n}}R_j-z}{2\sqrt{a\tau_s}}+(-1)^{\rho}\sqrt{\lambda\tau_s}\right]\right\}.
$$

Consideration of Various Degrees of Exothermy in Blocks of the Column

As a rule, blocks of a mass which are directly above each other are homogeneous as to concrete composition. This condition was kept in mind in the solutions of § 5-2: the parameters of exothermy q_0 and m were assumed identical for all blocks of the concrete column. However, in construction practice, cases do occur when for some reason as a mass is being constructed, concretes are used which differ both as to cement type and as to its content. Naturally, this leads to differences in heat liberation in the concrete blocks.

Suppose a mass consists of $\bar{n}$ blocks and a base. We assume that in the sth block ($s = 1, 2, \ldots, \bar{n}$) the intensity of heat liberation is determined by the expression

$$
q = q_{0s}e^{-m_s\tau},
$$

whereas in the remaining block

$$
q = q_{0c}e^{-m_c\tau}.
$$

The edge conditions of the problem are similar to those presented above.

Then the solution of the problem is equal to:

$$T = (T)_1 + (T)_2,$$

where $(T)_1$ is the solution presented in § 5-2 (with parameters of the heat liberation intensity function q_{0c} and m_c);
$(T)_2$ is the solution of the equation

$$\frac{\partial T}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T}{\partial z^2_{\bar{n}}} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{c\gamma}\left(q_{0s}\exp[-m_s t_s] - q_{0c}\exp[-m_c t_s]\right)\varepsilon(z_{\bar{n}})$$
$$(0 < z_{\bar{n}} < \infty,\ -L < x < L,\ -D < y < D;\ s = 1, 2, \ldots, n) \tag{5-30}$$

with zero initial and boundary conditions.

Here t_s is the "life" time of the block, equal to

$$t_s = \sum_{j=s+1}^{\bar{n}} \tau_j + \tau^{(\bar{n})};$$

$\varepsilon(z_{\bar{n}})$ is a unit function, equal to

$$\varepsilon(z_{\bar{n}}) = \begin{cases} 1 & \text{where } \sum\limits_{j=s+1}^{\bar{n}} R_j < z_{\bar{n}} < \sum\limits_{j=s}^{\bar{n}} R_j; \\ 0 & \text{where } 0 < z_{\bar{n}} < \sum\limits_{j=s+1}^{\bar{n}} R_j \ \text{и}\ \sum\limits_{j=s}^{\bar{n}} R_j < z_{\bar{n}} < \infty. \end{cases}$$

The solution

$$(T)_2 = T_s - T_c,$$

where

$$T_{k(k=s,c)} = \frac{q_{0k}}{c\gamma} \exp\left[-m_k \sum_{j=s+1}^{\bar{n}} \tau_j\right] \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} A_p B_r \times$$

$$\times \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \exp\left[-aK_{pr} \tau^{(\bar{n})}\right] \times$$

$$\times \sum_{\rho=1}^{4} \left\{ a_\rho \operatorname{erfc}\left[\frac{M_\delta(z_{\bar{n}})}{2\sqrt{a\tau^{(\bar{n})}}}\right] + b_\rho \exp\left[h_{z(\bar{n})}^2 a\tau^{(\bar{n})} + h_{z(\bar{n})} M_\delta(z_{\bar{n}})\right] \times\right.$$

$$\times \operatorname{erfc}\left[\frac{M_\delta(z_{\bar{n}})}{2\sqrt{a\tau^{(\bar{n})}}} + h_{z(\bar{n})}\sqrt{a\tau^{(\bar{n})}}\right] +$$

$$+ d_\rho I_{1/2} - \left(m_k - aK_{pr}, -\left(\frac{M_\delta(z_{\bar{n}})}{2\sqrt{a}}\right)^2, \tau^{(\bar{n})}\right) +$$

$$+ g_\rho I_{3/2}\left(m_k - aK_{pr}, -\left(\frac{M_\delta(z_{\bar{n}})}{2\sqrt{a}}\right)^2, \tau^{(\bar{n})}\right);$$

$$M_\delta(z_{\bar{n}}) = \sum_{j=s+\delta_\rho}^{\bar{n}} R_j + (-1)^\rho z_{\bar{n}};$$

$$\delta_\rho = \begin{cases} 1 \text{ where } \rho \leqslant 2; \\ 0 \text{ where } \rho > 2; \end{cases} \quad K_{pr} = \frac{\mu_p^2}{L^2} + \frac{\varkappa_r^2}{D^2}; \tag{5-31}$$

$I_{1/2}$ and $I_{3/2}$ are integrals, the value of which is determined by algorithm (5-21). The values of coefficients a_ρ, b_ρ, d_ρ and g_ρ are presented in Table 5-4.

The solution of the problem for the two-dimensional and one-dimensional cases is little different from the solution of (5-31). In the problem concerning the planar temperature field, summation should be performed only with respect to the roots μ_p, while K_{pr} should be replaced by

$$K_{pH} = \frac{\mu_p^2}{L^2} + \frac{H_{\bar{n}}}{a},$$

in the problem of the one-dimensional temperature field K_{pr} becomes $H_{\bar{n}}/a$.

Peculiarities of Calculation of Temperature Fields
Using Two-Dimensional and One-Dimensional Plans

To consider heat exchange from surfaces not included in the boundary conditions of two-dimensional and one-dimensional problems, i.e., with a third dimension for two-dimensional problems and with second and third dimensions

TABLE 5-4. VALUES OF COEFFICIENTS a_ρ, b_ρ, d_ρ and g_ρ

a_1	a_2	a_3	a_4
$\frac{1}{2(m_{\bullet} - aK_{pr})}$	$-a_1$	$-a_1$	a_1
b_1	b_2	b_3	b_4
0	$\frac{1}{h^2_{z(\bar{n})} a + m_t - aK_{pr}}$	0	$-b_2$
d_1	d_2	d_3	d_4
0	$\frac{1}{\sqrt{\pi}} h_{z(\bar{n})} \sqrt{a}\, b_2$	0	$-d_2$
g_1	g_2	g_3	g_4
$-\frac{1}{\sqrt{\pi}} a_1$	$\frac{1}{\sqrt{\pi}} (a_1 - b_2)$	$-g_1$	$-g_2$

for one-dimensional problems, differential equation for heat conductivity includes terms such as $H(T - \chi)$, corresponding to negative heat sources, so that the equation is written as

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + \frac{1}{c\gamma} q(\tau, T) - H(T - \chi), \tag{5-32}$$

where H is a parameter; χ is the ambient temperature on the surface not included in the boundary conditions of the problem.

The approach is similar in the theory of heat conductivity of rods[1], where the value of H is

$$H = \frac{\alpha P}{c\gamma S}.$$

Here P and S are the perimeter and area of the cross section of the rod; α is the heat transfer coefficient from the lateral surface of the rod; $c\gamma$ is the specific volumetric heat capacity.

[1]A rod refers to a body in which the dimensions of the cross section are significantly less than the length, so that the temperature field through the cross section can be assumed even.

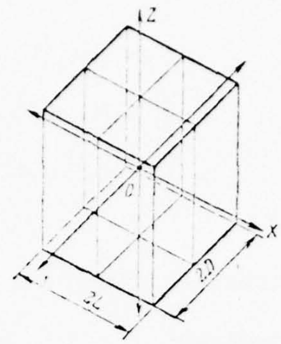

Figure 5-2. A Parallelepiped

In the case of concrete masses, due to the uneven distribution of temperature through the cross section of the body, quantity H is a parameter, the numerical value of which can be estimated on the basis of a simplified model of the phenomenon.

Let us study such models for certain calculation plans.

1. A concrete column. As the model, we select a parallelepiped (Figure 5-2).

The distribution of temperature in a symmetrically cooled parallelepiped (2L x 2D x 2R) with constant initial temperature T_0 and 0 temperature of the medium is determined by the expression

$$T(x, y, z, \tau) = T_0 X(\mathrm{Fo}_x, \mathrm{Bi}_x) Y(\mathrm{Fo}_y, \mathrm{Bi}_y) \times \times Z(\mathrm{Fo}_z, \mathrm{Bi}_z), \tag{5-33}$$

where X, Y, Z are the solutions of the corresponding one-dimensional problems:

$$X(\mathrm{Fo}_x, \mathrm{Bi}_x) = \sum_{n=1}^{\infty} A_n \cos \mu_n \frac{x}{L} e^{-\mu_n^2 \mathrm{Fo}_x};$$

$$Y(\mathrm{Fo}_y, \mathrm{Bi}_y) = \sum_{m=1}^{\infty} B_m \cos \varkappa_m \frac{y}{D} e^{-\varkappa_m^2 \mathrm{Fo}_y};$$

$$Z(\mathrm{Fo}_z, \mathrm{Bi}_z) = \sum_{r=1}^{\infty} C_r \cos \eta_r \frac{z}{R} e^{-\eta_r^2 \mathrm{Fo}_z};$$

μ_n, κ_m, η_r are the roots of the characteristic equations

$$\operatorname{ctg}\mu_n = \frac{\mu_n}{\mathrm{Bi}_x},\quad \operatorname{ctg}\varkappa_m = \frac{\varkappa_m}{\mathrm{Bi}_y},\quad \operatorname{ctg}\eta_r = \frac{\eta_r}{\mathrm{Bi}_z};$$

$\mathrm{Fo}_x = \frac{a\tau}{L^2}$, $\mathrm{Fo}_y = \frac{a\tau}{D^2}$, $\mathrm{Fo}_z = \frac{a\tau}{R^2}$ is the Fourier criterion;

$\mathrm{Bi}_x = \frac{\alpha_x L}{\lambda}$, $\mathrm{Bi}_y = \frac{\alpha_y D}{\lambda}$, $\mathrm{Bi}_z = \frac{\alpha_z R}{\lambda}$ is the Biot criterion;

$\alpha_{\xi(\xi=x,\ y,\ z)}$ is the heat transfer factor.

On the axis of the parallelepiped

$$T(0,\ 0,\ z,\ \tau) = T_0\,[X][Y]\,Z(\mathrm{Fo}_z,\ \mathrm{Bi}_z),$$

where

$$[X] = \sum_{n=1}^{\infty} A_n e^{-\mu_n^2 \mathrm{Fo}_x};$$

$$[Y] = \sum_{m=1}^{\infty} B_m e^{-\varkappa_m^2 \mathrm{Fo}_y}.$$

On the other hand, analyzing the problem as a one-dimensional problem (along the OZ axis) with negative heat sources of form HT, for the temperature on the axis of the parallelepiped we produce the expression

$$T(z,\ \tau) = T_0 Z(\mathrm{Fo}_z,\ \mathrm{Bi}_z)\exp\left[-\int_0^{\tau} H\,d\tau\right],$$

where H = H(τ).

Consequently, in order to determine parameter H in the one-dimensional problem we have the equation

$$[X][Y] = \exp\left[-\int_0^{\tau} H\,d\tau\right]. \qquad (5\text{-}34')$$

Let us introduce the new variable

$$\mathrm{Fo}_{xy} = \left(\frac{1}{L^2} + \frac{1}{D^2}\right) a\tau$$

and assume

$$H^*_{\text{об}} = \frac{H}{a\left(\frac{1}{L^2} + \frac{1}{D^2}\right)}.$$

Then equation (5-34') is rewritten as

$$[X][Y] = \exp\left[-\int_0^{\mathrm{Fo}_{\text{об}}} H^*_{\text{об}}\, d\,\mathrm{Fo}_{\text{об}}\right]. \tag{5-34}$$

The temperature on plane YOZ is:

$$T(0,\, y,\, z,\, \tau) = T_0[X]YZ.$$

Based on the two-dimensional equation for heat conductivity with a type HT negative heat source, we find

$$T(y,\, z,\, \tau) = T_0 YZ \exp\left[-\int_0^{\tau} H(\tau)\, d\tau\right].$$

From here, to determine parameter H in the two-dimensional problem we produce the equation

$$[X] = \exp\left[-\int_0^{\tau} H(\tau)\, d\tau\right] \tag{5-35'}$$

or

$$[X] = \exp\left[-\int_0^{\mathrm{Fo}_x} H^*_x\, d\,\mathrm{Fo}_x\right], \tag{5-35}$$

where

$$H^*_x = \frac{HL^2}{a}.$$

An equation similar to (5-35) obtains upon transition from the two-dimensional problem to the one-dimensional.

If we assume that $H(\tau)$ is a piecewise-constant function of time, then

$$\int_0^\tau H(\tau)\,d\tau = \int_0^{\tau_1} H_1 d\tau + \int_{\tau_1}^{\tau_2} H_2 d\tau + \ldots + \int_{\tau_l}^{\tau} H_l d\tau =$$

$$= H_1\tau_1 + H_2(\tau_2 - \tau_1) + \ldots + H_l(\tau - \tau_l).$$

Then the values of H_k $(k = 1, 2, \ldots, \ell)$ can be easily established from equations (5-34) and (5-35) by successive transition from one time interval to the next.

2. A concrete column constructed "in a space." The model is shown in Figure 5-3.

The temperature field of a symmetrically cooled unlimited prism with initial temperature T_0 in area $(-L \leq x \leq L)$ and zero temperature in the remaining portion $(-\infty < x < -L,\ L < x < \infty)$ and with zero ambient temperature is equal to:

$$T(x, y, z, \tau) = T_0 X(\mathrm{Fo}_x)\, Y(\mathrm{Fo}_y, \mathrm{Bi}_y)\, Z(\mathrm{Fo}_z, \mathrm{Bi}_z),$$

where

$$X(\mathrm{Fo}_x) = 1 - \frac{1}{2}\operatorname{erfc}\left[\frac{1 - \frac{x}{L}}{2\sqrt{\mathrm{Fo}_x}}\right] - \frac{1}{2}\operatorname{erfc}\left[\frac{1 + \frac{x}{L}}{2\sqrt{\mathrm{Fo}_x}}\right];\quad \mathrm{Fo}_x = \frac{a\tau}{L^2}.$$

The functions $Y(\mathrm{Fo}_y, \mathrm{Bi}_u)$ and $Z(\mathrm{Fo}_z, \mathrm{Bi}_z)$ have the same values as in the preceding section 1. On the OZ axis, the temperature is

$$T(0, 0, z, \tau) = [X][Y] Z(\mathrm{Fo}_z, \mathrm{Bi}_z)$$

in the three-dimensional problem and

$$T(z, \tau) = Z(\mathrm{Fo}_z, \mathrm{Bi}_z) \exp\left[-\int_0^\tau H(\tau)\,d\tau\right]$$

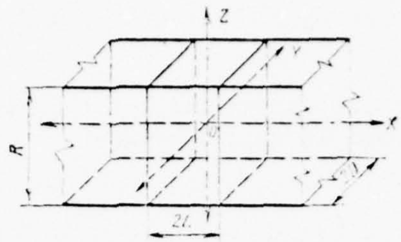

Figure 5-3. Model of Concrete Column Constructed "In a Space"

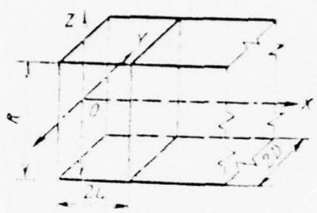

Figure 5-4. Model of Concrete Column Adjacent to the Side of a Canyon

in the one dimensional problem.

Thus, parameter H in the one-dimensional problem is defined by the equation

$$[X][Y] = \exp\left[-\int_0^{\tau} H(\tau)\,d\tau\right]$$

where

$$[X] = 1 - \operatorname{erfc}\left[\frac{1}{2\sqrt{\mathrm{Fo}_x}}\right] = \operatorname{erf}\left[\frac{1}{2\sqrt{\mathrm{Fo}_x}}\right].$$

In the two-dimensional problem we have the equation

$$[X] = \exp\left[-\int_0^{\tau} H(\tau)\,d\tau\right].$$

3. A concrete column adjacent to the side of a canyon. The model which we use is a semilimited prism (Figure 5-4) with an initial temperature of T_0 where $0 \leq x \leq 2L$, and zero where $2L < x < \infty$, and with zero ambient temperature.

The basic relationships for determination of parameter $H(\tau)$ are the same as in the previous cases, but we should assume:

$$X = 1 + \exp\left[\mathrm{Bi}_x^2 \mathrm{Fo}_x + \mathrm{Bi}_x \frac{x}{L}\right] \mathrm{erfc}\left[\frac{x/L}{2\sqrt{\mathrm{Fo}_x}} + \mathrm{Bi}_x\sqrt{\mathrm{Fo}_x}\right] - \exp\left[\mathrm{Bi}_x^2 \mathrm{Fo}_x + \mathrm{Bi}_x\left(2 + \frac{x}{L}\right)\right] \mathrm{erfc}\left[\frac{2 + x/L}{2\sqrt{\mathrm{Fo}_x}} + \mathrm{Bi}_x\sqrt{\mathrm{Fo}_x}\right] - \mathrm{erfc}\left[\frac{x/L}{2\sqrt{\mathrm{Fo}_x}}\right] + \frac{1}{2}\mathrm{erfc}\left[\frac{2 + x/L}{2\sqrt{\mathrm{Fo}_x}}\right] - \frac{1}{2}\mathrm{erfc}\left[\frac{2 - x/L}{2\sqrt{\mathrm{Fo}_x}}\right].$$

The coordinates of the axial line of the block are (L, 0, z) and therefore

$$|X| = 1 + \exp\left[\mathrm{Bi}_x^2 \mathrm{Fo}_x + \mathrm{Bi}_x\right] \mathrm{erfc}\left[\frac{1}{2\sqrt{\mathrm{Fo}_x}} + \mathrm{Bi}_x\sqrt{\mathrm{Fo}_x}\right] - \exp\left[\mathrm{Bi}_x^2 \mathrm{Fo}_x + 3\,\mathrm{Bi}_x\right] \mathrm{erfc}\left[\frac{3}{2\sqrt{\mathrm{Fo}_x}} + \mathrm{Bi}_x\sqrt{\mathrm{Fo}_x}\right] - \frac{3}{2}\mathrm{erfc}\left[\frac{1}{2\sqrt{\mathrm{Fo}_x}}\right] + \frac{1}{2}\mathrm{erfc}\left[\frac{3}{2\sqrt{\mathrm{Fo}_x}}\right];$$

The relationships presented can be used also in the case of concrete masses. This statement is based on the following considerations.

Heat liberation in the mass is basically completed in the first 20 to 30 days of curing of the concrete. Under these conditions, the temperature field after 2000 hours and more following pouring of the concrete is little dependent on detailed description of the process of heat liberation, and the exothermy of the concrete can be considered by introducing the equivalent mean temperature through the volume.

As concerns variable ambient temperature χ, it is included in the differential equation in the form of the term $H(T - \chi)$.

Special calculations have confirmed the expediency of introducing these terms into the differential equations of the planar and linear problems. This is illustrated, for example, by the data of Table 5-5, where we present the results of calculation of temperature along the axis of a planar concrete mass at various moments in time after the beginning of construction.

Concreting block height is 3 m, the interval of overlap of blocks neighboring in height is 240 hr, the width of a column is 15 m, the total height of a column is 18 m. The ambient temperature is constant on the horizontal surface (concrete poured beneath a tent) and changes harmonically on the side surface. The initial temperatures of the concrete mixture and base differ. In the column "model of two-dimensional body" we present values of temperatures produced by solution of the two-dimensional equation for heat conductivity ignoring thermal losses through the third dimension; the model of a one-dimensional body was analyzed in two versions: 1) on the assumption that the lateral surface did not influence the temperature along the axis of the column; and 2) considering heat losses from the second (through the width of the column) dimension. In the last case, we used the following numerical values of parameter H, determined for the condition of transition from a two-dimensional problem to a one-dimensional problem:

Time Interval, hr	Values of H, C/(hr)
0—720	0
720—1 440	$0,83 \cdot 10^{-5}$
1 440—2 880	$0,361 \cdot 10^{-4}$
2 880—4 320	$0,576 \cdot 10^{-4}$
4 320—5 760	$0,701 \cdot 10^{-4}$
5 760—7 200	$0,729 \cdot 10^{-4}$
7 200—9 840	$0,731 \cdot 10^{-4}$

TABLE 5-5. RESULTS OF CALCULATION OF TEMPERATURE ON AXIS OF A PLANAR CONCRETE COLUMN WITH VARIOUS MODELS OF THE BODY

1	2	3	4	2	3	4	2	3	4
	τ=2 640 hr			τ=4 800 hr			τ=9 840 hr		
0	6,80	6,81	6,93	5,93	5,89	6,08	5,35	5,26	5,64
1,5	21,63	21,64	21,88	13,88	13,68	15,36	8,22	7,86	11,22
3,0	32,03	32,04	32,69	20,85	20,62	23,55	10,97	[illegible]	[illegible]
4,5	37,52	37,24	38,18	26,32	26,03	29,99	13,46	12,45	[illegible]
6,0	39,47	38,95	40,30	29,92	29,67	34,41	15,52	[illegible]	[illegible]
7,5	39,72	39,03	40,90	31,81	31,63	36,93	17,03	15,88	27,57
9,0	39,33	38,58	40,56	32,29	32,20	37,95	17,86	16,81	[illegible]
10,5	38,51	37,83	40,06	31,65	31,98	37,46	18,19	17,23	[illegible]
12,0	37,02	36,59	39,80	30,21	30,28	37,90	17,88	17,17	[illegible]
13,5	34,60	34,49	36,60	27,83	28,17	33,69	16,88	[illegible]	[illegible]
15,0	31,05	31,22	32,95	24,88	25,56	30,37	15,97	[illegible]	26,73
16,5	26,40	26,82	28,15	21,61	22,63	26,64	14,61	15,45	21,43
18,0	21,09	21,74	22,53	18,08	19,71	22,36	13,20	14,87	21,80

Key: 1, Distance from Upper Horizontal Surface, m; 2, Model of Two-dimensional Body; 3, Model of One-Dimensional Body Considering H; 4, Model of One-Dimensional Body Ignoring H

The data of Table 5-5 show how important in calculation of temperature fields of columnar masses (after 2000 hr and more following beginning of construction) it is to consider heat losses from surfaces not included in the

problem. Furthermore, they confirm the method suggested for determination of the numerical values of parameter H.

Comments

1. We have just analyzed symmetrical heat transfer from surfaces not included in the boundary conditions of the problem. The results produced, however, can be extended to the case of asymmetrical heat transfer as well: in place of one term in the differential equation of heat conductivity, we must introduce the sum

$$\sum_{j=1}^{l} \beta_j H_j (T - \chi_j),$$

where $\ell = 4$ upon transition from the three-dimensional problem to a one-dimensional problem; $\ell = 2$ upon transition from the two-dimensional problem to the one-dimensional problem; β_j is the portion of the lateral surface for which the values of parameter H_j and ambient temperature χ_j are characteristic. The numerical values of H_j here are established on the basis of the considerations presented above.

2. When terms such as $H(T - \chi)$ are introduced to the differential equation for heat conductivity, it is possible, with accuracy sufficient for engineering practice, to use a one-dimensional plan to calculate the temperature on the axis of the columnar mass and, using a two-dimensional plan -- the temperature in its central vertical cross section.

Suitability of Model of Block-by-Block Growth of a Column

The statement of the problem and its solution presented in § 5-2 precisely reflect the conditions of construction of a mass, the base of which is an "old" concrete column. Calculation studies have shown that the model used for block-by-block growth of the column satisfactorily describes the thermal mode of a concrete mass constructed on a rocky base as well. This is confirmed, for example, by the data of Table 6, where we present the results of calculation of the planar temperature field of a concrete mass near the base, produced from the analytic solution presented in § 5-2 and the finite difference solution described in § 5-5. This last solution more completely considers the geometry of the mass, and therefore the results of calculation using the finite difference solution were accepted as the standards.

The following were assumed: height of concrete block R = 9 m, width of block 2L = 14 m, heat-physical characteristics $a = 3 \cdot 10^{-3}$ m²/hr, $\lambda = 1.7$ kcal/(m·hr·C); parameters of heat liberation intensity function in the concrete $q_0/c\gamma = 0.4$ C/hr, m = 0.01 1/hr, initial temperature of concrete 10 c; initial temperature of base 4 C; ambient temperature 10 C; heat transfer coefficient on horizontal surface of concrete 8.5 kcal/(m²·hr·C), on the

TABLE 5-6. COMPARISON OF CALCULATED VALUES OF TEMPERATURES, C, OF CONCRETE MASS NEAR BASE

1	2 Координаты, м		3 Температура, °С / 4 Время от начала возведения, ч					
			240		480		720	
	z	x	5 Расчет по аналитическим решениям	6 Расчет по конечно-разностным решениям	5 Расчет по аналитическим решениям	6 Расчет по конечно-разностным решениям	5 Расчет по аналитическим решениям	6 Расчет по конечно-разностным решениям
(11)	0	7,0	8,3	—	3,0	—	0,2	—
(12)	0	5,6	21,8	22,1	19,1	18,9	16,2	15,8
(13)	0	4,2	25,1	25,3	25,3	25,3	23,0	23,6
(14)	0	2,8	24,9	25,6	26,6	26,7	26,2	26,3
(21)	1,8	7,0	8,3	7,8	3,1	1,4	0,3	0,3
(22)	1,8	5,6	39,2	38,5	31,2	30,8	24,2	24,0
(23)	1,8	4,2	44,3	44,3	41,8	41,6	36,2	36,4
(24)	1,8	2,8	44,6	44,7	44,0	44,0	40,8	40,8
(31)	3,6	7,0	8,8	8,2	4,0	4,0	1,2	1,4
(32)	3,6	5,6	40,7	39,8	34,6	34,3	27,9	27,8
(33)	3,6	4,2	46,0	45,8	46,4	46,1	42,0	41,9
(34)	3,6	2,8	46,3	46,3	48,8	48,8	46,9	46,8
(41)	5,4	7,0	8,8	8,2	3,9	4,2	0,8	1,9
(42)	5,4	5,6	40,6	39,8	34,0	33,8	26,5	26,6
(43)	5,4	4,2	46,0	45,8	45,7	45,5	40,2	40,3
(44)	5,4	2,8	46,3	46,3	48,1	48,2	44,9	45,1

Key: 1, Number; 2, Coordinates, m; 3, Temperature, C; 4, Time from Beginning of Construction, hr; 5, Analytic Solution; 6, Finite Difference Solution

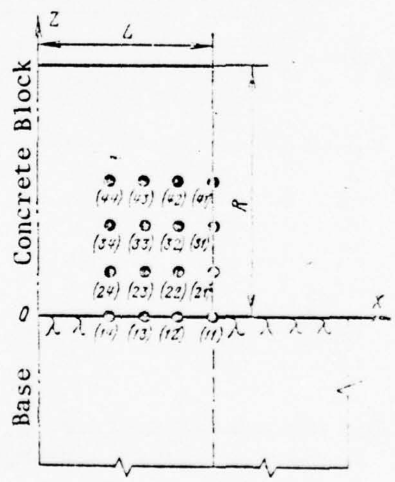

Figure 5-5. Placement of Points in Mass Used to Compare Calculated and Actual Temperature

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lateral surfaces of the concrete -- 3.4 kcal/($m^2 \cdot hr \cdot C$), on the horizontal surface of the base -- 20 kcal/($m^2 \cdot hr \cdot C$)[1]. The placement of points used to compare the temperature, calculated in standard, can be seen from Figure 5-5.

Limits of Applicability of Assumption of Instantaneous Placement of Blocks

In § 5-1 of this chapter, we indicated the expediency of modeling the process of construction of a mass by assuming instantaneous placement of concrete blocks. In this case, for short blocks, we can produce satisfactory results with a significantly smaller volume of calculation than that required if we consider the layer-by-layer pouring of the concrete.

It is therefore interesting to establish the values of block height for which the assumption of instantaneous placement can be used.

In Table 5-7 we present the calculated values of temperature in a block 3 m high, produced on the basis of the model of layer-by-layer pouring (layer height 50 cm, intervals between layer overlap 2 hr) and instantaneous placement.

TABLE 5-7. RESULTS OF CALCULATION OF TEMPERATURE, C, IN A CONCRETE BLOCK 3 m HIGH USING THE MODEL OF LAYER-BY-LAYER POURING AND INSTANTANEOUS PLACEMENT

1 Расстояние от верхней горизонтальной поверхности, м		2 Время после укладки последнего слоя, τ_k, ч			
		63		120	
		3 Модель послойного бетонирования (реальная картина)	4 Модель мгновенной укладки	3 Модель послойного бетонирования	4 Модель мгновенной укладки
5 Бетонный блок	0	16,6	16,5	19,7	19,6
	0,6	26,2	25,9	32,1	31,9
	1,2	28,5	27,8	36,5	36,3
	1,8	28,7	27,7	35,4	36,2
	2,4	26,5	25,6	32,9	31,9
	3,0	16,8	16,0	21,3	21,0
6 Основание	3,6	7,0	6,5	10,5	10,0
	4,2	4,4	4,3	5,3	5,5

Key: 1, Distance from Top Horizontal Surface, m; 2, Time After Pouring of Previous Layer, τ_k, hr; 3, Layer-by-Layer Pouring (Actual Picture); 4, Instantaneous Placement; 5, Concrete Block; 6, Base

[1] The heat transfer coefficient on the horizontal surface of the base was used in calculation of the standard temperature of the mass based on the finite difference solutions of § 5-5.

Figure 5-6 shows the dynamics of formation of the temperature field of the block in the process of layer-by-layer pouring.

Table 5-8 and Figure 5-7 show the results of calculation of the temperature in a block 6 m high. The height of the pouring layer is 75 cm, intervals of layer overlap 3 hr. Since the concreting time of a 6 meter block is considerable (τ_B = 24 hr), Table 5-8 includes an additional column with the results of calculation of the temperature at moments in time shifted by $1/2\ \tau_B$ = 12 hr.

TABLE 5-8. RESULTS OF CALCULATION OF TEMPERATURE, C, IN A CONCRETE BLOCK 6 m HIGH ON THE BASIS OF THE MODELS OF LAYER-BY-LAYER POURING AND INSTANTANEOUS PLACEMENT

1 Расстояние от верхней горизонтальной поверхности, м	2 Время после укладки последнего слоя $\tau_к$, ч					
	3 Модель послойного бетонирования	4 Модель мгновенной укладки		3 Модель послойного бетонирования	4 Модель мгновенной укладки	
		$\tau_к = 18$ hr	$\tau_к = 18$ hr $+ 1/2\tau_в$		$\tau_к = 144$ hr	$\tau_к = 144$ hr $+ 1/2\tau_в$
0	11,3	11,3	11,6	13,7	11,9	11,3
1,2	25,7	25,1	27,7	37,6	37,2	37,7
2,4	26,9	25,2	28,3	40,9	40,4	41,5
3,6	28,0	25,2	28,3	41,3	40,5	41,6
4,8	28,6	25,1	28,0	39,1	38,5	39,2
6,0	17,3	15,1	16,5	23,4	22,8	23,5

Key: 1, Distance from Top Horizontal Surface, m; 2, Time After Placement of Previous Block, τ_k, hr; 3, Layer-by-Layer Concreting Model; 4, Instantaneous Placement Model

A special analysis based on calculation of the type presented above has shown that with the rates of construction of masses used in hydraulic engineering construction, blocks up to 3-6 m in height can be considered short and the approximation of instantaneous placement used for them.

Calculation of Temperature Field of a Concrete Complex Consisting of Blocks Neighboring in Plan

The area studied is schematically illustrated in Figure 5-8. For greater generality, we assume that the blocks are constructed at different times. Heat liberation occurs in the blocks, each block being characterized by its own values of heat liberation intensity function q_{0i} and m_i (i = 1, 2). The heat-physical characteristics of the blocks and the base are identical. The time count begins at the moment of placement of block 2. The problem is then formulated as follows.

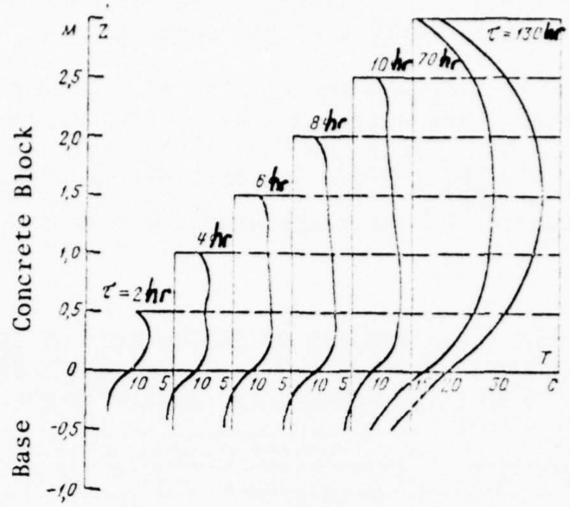

Figure 5-6. Dynamics of Formation of Temperature Field of Block in the Process of Layer-by-Layer Pouring (Layer Height 50 cm, Coverage Interval 2 hr, Block Height 3 m)

The differential equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{c\gamma} q_{01}\varepsilon(x)\,\omega(z)\exp[-m_1(t_1+\tau)] +$$
$$+\frac{1}{c\gamma} q_{02}[1-\varepsilon(x)]\,\omega(z)\exp[-m_2\tau]$$
$$(0 < z < \infty,\ -L < x < L,\ -D < y < D,\ \tau > 0). \tag{5-36}$$

Initial condition

$$T(z, x, y, 0) = \begin{cases} \Phi_b(z, x, y) \text{ where } 0 < z < R, \\ \Phi_0(z, x, y) \text{ where } R < z < \infty; \end{cases}$$
$$\Phi_b(z, x, y) = \begin{cases} f_b(z, x, y) \text{ where } -L_1 < x < L_1,\ -D < y < D, \\ T_0 \text{ where } -L < x < -L_1 \text{ и } L_1 < x < L,\ -D < y < D. \end{cases}$$

Boundary conditions are similar to (5-24), but we must consider the ambient

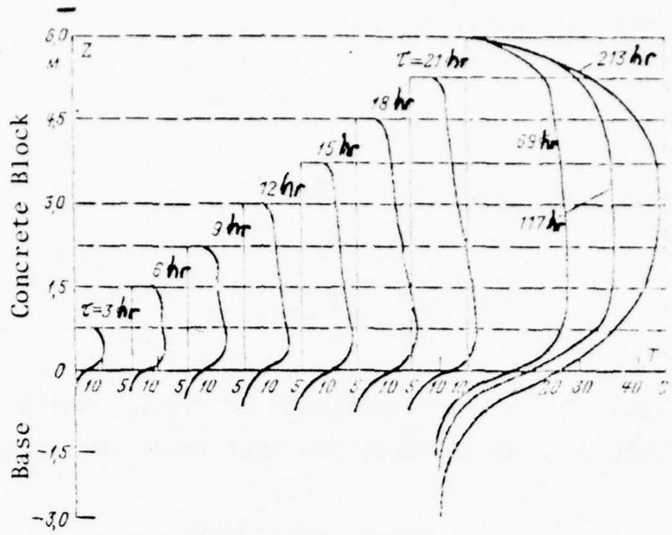

Figure 5-7. Dynamics of Formation of Temperature Field in a Block During Layer-by-Layer Pouring (Layer Height 75 cm, Layer Overlap Interval 3 hr, Block Height 6 m)

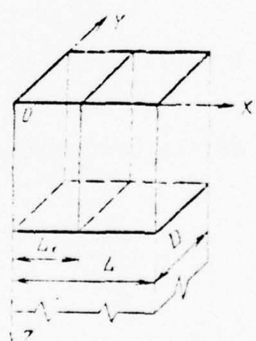

Figure 5-8. Diagram of Calculation Area in the Problem Concerning the Temperature Field of a Complex Consisting of Blocks Neighboring in Plan

temperature on the horizontal surface

$$\varphi(\tau) = T_1, \tag{5-37}$$

and on the vertical surfaces

$$\Psi(z, \tau) = \begin{cases} T_2 & \text{where } 0 < z < R, \\ T_{cu} & \text{where } R < z < \infty. \end{cases}$$

In equation (5-36), T_1 is the "life" time of block 1 until the moment of placement of block 2, $\varepsilon(x)$ and $\omega(z)$ are unit functions equal to

$$\varepsilon(x) = \begin{cases} 1 & \text{where } -L_1 < x < L_1, \\ 0 & \text{where } -L < x < -L_1 \text{ и } L_1 < x < L; \end{cases} \tag{5-37'}$$

$$\omega(z) = \begin{cases} 1 & \text{where } 0 < z < R, \\ 0 & \text{where } R < z < \infty. \end{cases}$$

We assume

$$T = (T)_1 + (T)_2,$$

where $(T)_1$ is the solution of the homogeneous heat conductivity equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

with zero initial condition and heterogeneous boundary conditions, it is easily determined from the results of the previous paragraph that $(T)_2$ is the solution of differential equation (5-36) with initial condition (5-37) and homogeneous (with zero ambient temperature) boundary conditions.

Without discussing the method of solution of the problem for the temperature function $(T)_2$, let us write the final result:

$$
\begin{aligned}
(T)_2 = {} & \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{\| U_0 \|^2 \| V_0 \|^2} \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \exp[-aK_{pr}\tau] \times \\
& \times \left\{ \int_0^R \int_0^D \int_0^{L_1} f_0(\eta, x, y) \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} V(z, \eta, \tau)\, d\eta\, dx\, dy + \right. \\
& \left. + \int_0^{\infty} \int_0^D \int_0^L \Phi_0(\eta, x, y) \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} V(z, \eta, \tau)\, d\eta\, dx\, dy \right\} + \\
& + \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} B_r \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \exp[-aK_{pr}\tau] \times \\
& \times \left\{ \left(A_p - \frac{L}{\mu_p \| U_0 \|^2} \sin \mu_p \frac{L_1}{L} \right) \left[T_0 J^{(0)}(z, \tau) + \frac{q_{02}}{c\gamma} J_{pr}^{(2)}(z, \tau) \right] + \right. \\
& \left. + \frac{q_{01}}{c\gamma} \frac{L}{\mu_p \| U_0 \|^2} \sin \mu_p \frac{L_1}{L} \exp[-m_1 t_1] J_{pr}^{(2)}(z, \tau) \right\}.
\end{aligned}
\tag{5-38}
$$

where

$$
\begin{aligned}
V(z, \eta, \tau) = {} & \frac{1}{2\sqrt{\pi a\tau}} \left(\exp\left[-\frac{(z-\eta)^2}{4a\tau} \right] + \exp\left[-\frac{(z+\eta)^2}{4a\tau} \right] \right) - \\
& - h_z \exp[h_z^2 a\tau + h_z(z+\eta)] \operatorname{erfc}\left[\frac{z+\eta}{2\sqrt{a\tau}} + h_z\sqrt{a\tau} \right]; \\
J^{(0)}(z, \tau) = {} & \operatorname{erf}\left[\frac{z}{2\sqrt{a\tau}} \right] - \frac{1}{2} \operatorname{erfc}\left[\frac{R-z}{2\sqrt{a\tau}} \right] + \\
& + \frac{1}{2} \operatorname{erfc}\left[\frac{R+z}{2\sqrt{a\tau}} \right] + \exp[h_z^2 a\tau + h_z z] \operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau}} + \right. \\
& \left. + h_z\sqrt{a\tau} \right] - \exp[h_z^2 a\tau + h_z(R+z)] \operatorname{erfc}\left[\frac{R+z}{2\sqrt{a\tau}} + h_z\sqrt{a\tau} \right];
\end{aligned}
\tag{5-39}
$$

$$f_{pr}^{(i)}(z,\tau)=\frac{1}{(m_i-aK_{pr})}\left(\operatorname{erf}\left[\frac{z}{2\sqrt{a\tau}}\right]-\frac{1}{2}\operatorname{erfc}\left[\frac{R-z}{2\sqrt{a\tau}}\right]+\right.$$

$$\left.+\frac{1}{2}\operatorname{erfc}\left[\frac{R+z}{2\sqrt{a\tau}}\right]\right)+\frac{1}{(h_z^2a+m_i-aK_{pr})}\exp[h_z^2a\tau+h_zz]\times$$

$$\times\left(\operatorname{erfc}\left[\frac{z}{2\sqrt{a\tau}}+h_z\sqrt{a\tau}\right]-\exp[h_zR]\operatorname{erfc}\left[\frac{R+z}{2\sqrt{a\tau}}+h_z\sqrt{a\tau}\right]+\right.$$

$$+\frac{h_z^2a}{\sqrt{\pi}(m_i-aK_{pr})(h_z^2a+m_i-aK_{pr})}I_{3/2}\left(m_i-aK_{pr},-\left(\frac{z}{2\sqrt{a}}\right)^2,\tau\right)-$$

$$-\frac{h_z^2a-m_i+aK_{pr}}{2\sqrt{\pi}(m_i-aK_{pr})(h_z^2a+m_i-aK_{pr})}I_{3/2}\left(m_i-aK_{pr},-\left(\frac{R+z}{2\sqrt{a}}\right)^2,\tau\right)+$$

$$+\frac{1}{2\sqrt{\pi}(m_i-aK_{pr})}I_{3/2}\left(m_i-aK_{pr},-\left(\frac{R-z}{2\sqrt{a}}\right)^2,\tau\right)+$$

$$+\frac{h_z\sqrt{a}}{\sqrt{\pi}(h_z^2a+m_i-aK_{pr})}\left\{I_{1/2}\left(m_i-aK_{pr},-\left(\frac{z}{2\sqrt{a}}\right)^2,\tau\right)-\right.$$

$$\left.-I_{1/2}\left(m_i-aK_{pr},-\left(\frac{R+z}{2\sqrt{a}}\right)^2,\tau\right)\right\}\quad(i=1,2);$$

$$\|U_0\|^2=\frac{L}{2}\frac{Bi_x^2+Bi_x+\mu_p^2}{Bi_x^2+\mu_p^2};\quad\|V_0\|^2=\frac{D}{2}\frac{Bi_y^2+Bi_y+\varkappa_r^2}{Bi_y^2+\varkappa_r^2}.\tag{5-40}$$

In order to calculate the triple integrals of the first sum, it is convenient to approximate the functions $f_b(z, x, y)$ and $\Phi_0(x, z, y)$ with polynomials and use recurrent relationships, presented in the previous section.

The solution to (5-38) presented above is easily extended to the case of any number of neighboring blocks equal in height and width, located along a single line, but poured at different times.

The transition from the three-dimensional problem to the planar problem is obvious.

Calculations of the Temperature Field of a Concrete Mass with Broad Cooling Seams after They are Filled

The system of breaking up hydraulic engineering structures into columnar masses with broad concreted cooling seams has been used both in the USSR and abroad. The width of the cooling seams is primarily determined by the method of performance of the work and the distance between seams, and varies between 0.6 and 2.4 m. The seams are usually concreted after cooling of the main concrete structure. For example, in construction of the Mamakan Dam,

most of the longitudinal seams were 1.3-1.5 m in width and were filled by pouring of concrete mixture into the seams without cementation.

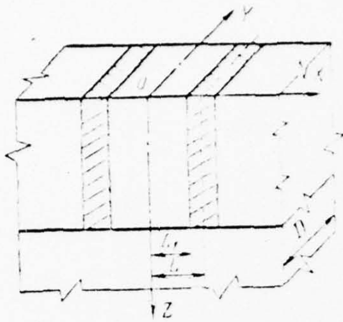

Figure 5-9. Diagram of Calculation Area in the Problem of the Temperature Field of a Mass with Broad Cooling Seams

Therefore, there is some interest in the method of calculation of the temperature mode of a mass after filling of a broad seam between columns.

The primary element in such a mass is the area schematically illustrated in Figure 5-9.

Formulation of the problem. The differential equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{c\gamma} q_{0\varepsilon}(x)\,\omega(z)\exp[-m\tau]$$
$$(0 < z < \infty,\ 0 < x < L,\ 0 < y < D,\ \tau > 0). \qquad (5\text{-}41)$$

The initial conditions

$$T(z, x, y, 0) = \begin{cases} \Phi_6(z, x, y) & \text{where } 0 < z < R, \\ \Phi_0(z, x, y) & \text{where } R < z < \infty; \end{cases}$$
$$\Phi_6(z, x, y) = \begin{cases} f_6(z, x, y) & \text{where } 0 < x < L_1, \\ T_0 & \text{where } L_1 < x < L. \end{cases}$$

The boundary conditions

$$\frac{\partial T(0, x, y, \tau)}{\partial z} = -h_z [T_1 - T(0, x, y, \tau)];$$

$$\frac{\partial T(\infty, x, y, \tau)}{\partial z} = 0;$$

$$\frac{\partial T(z, L, y, \tau)}{\partial x} = \frac{\partial T(z, 0, y, \tau)}{\partial x} = 0;$$

$$\frac{\partial T(z, x, D, \tau)}{\partial y} = h_y [\Psi(z) - T(z, x, D, \tau)];$$

$$\frac{\partial T(z, x, 0, \tau)}{\partial y} = 0,$$

where

$$\Psi(z) = \begin{cases} T_2 & \text{where } 0 < z < R, \\ T_{CK} & \text{where } R < z < \infty; \end{cases}$$

$\varepsilon(x)$ and $\omega(z)$ are unit functions defined by (5-37).

The solution

$$T = (T)_1 + (T)_2, \tag{5-42}$$

where $(T)_1$ is the solution of the two-dimensional (in coordinates (z, y)) problem for a mass consisting of one block and a base, without heat liberation, with zero initial temperature and ambient temperature T_1 on the horizontal surface and T_2 and T^{CK} on the vertical surface, presented in the previous section [see (5-9)], (in the last formula, we should assume $\overline{n} = 1$, $q_0 = 0$, $\Phi(0)(z, y) = 0$, $H_n = 0$);

$$(T)_2 = \sum_{r=1}^{\infty} \frac{1}{\|V_0\|^2} \cos \varkappa_r \frac{y}{D} \exp\left[-\varkappa_r^2 \frac{a\tau}{D}\right] \times$$

$$\times \left\{ \int_0^R \int_0^D \int_0^{L_1} f_0(\eta, x, y) \cos \varkappa_r \frac{y}{D} V(z, \eta, \tau)\, d\eta\, dx\, dy + \right.$$

$$\left. + \int_R^{\infty} \int_0^D \int_0^L \Phi_0(\eta, x, y) \cos \varkappa_r \frac{y}{D} V(z, \eta, \tau)\, d\eta\, dx\, dy \right\} +$$

$$+ \left(1 - \frac{L_1}{L}\right) \sum_{r=1}^{\infty} B_r \cos \varkappa_r \frac{y}{D} \exp\left[-\varkappa_r^2 \frac{a\tau}{D^2}\right] \left(T_0 J^{(0)}(z, \tau) + \right.$$

$$+\frac{q_0}{c\gamma}J_r^{(2)}(z,\tau)\Big)+\frac{L}{2}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{\|V_0\|^2}\cos n\pi\frac{x}{L}\cos\varkappa_r\frac{y}{D}\times$$

$$\times\exp[-aK_{nr}\tau]\frac{1}{2\sqrt{\pi a\tau}}\Big\{\int_0^R\int_0^D\int_0^{L_1}f_0(\eta,x,y)\cos n\pi\frac{x}{L}\times$$

$$\times\cos\varkappa_r\frac{y}{D}V(z,\eta,\tau)\,d\eta\,dx\,dy+$$

$$+\int_R^{\infty}\int_0^D\int_0^L\Phi_0(\eta,x,y)\cos n\pi\frac{x}{L}\cos\varkappa_r\frac{y}{D}V(z,\eta,\tau)\,d\eta\,dx\,dy\Big\}-$$

$$-2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{n\pi}B_r\sin n\pi\frac{L_1}{L}\exp[-aK_{nr}\tau]\times$$

$$\times\Big(T_0J^{(0)}(z,\tau)+\frac{q_0}{c\gamma}J_{nr}^{(2)}(z,\tau)\Big). \tag{5-43}$$

Here $\varkappa_r$, $\|V_0\|^2$, $V(z,\eta,\tau)$, $J^{(0)}(z,\tau)$ have the same values as earlier;

$$K_{nr}=\frac{n^2\pi^2}{L^2}+\frac{\varkappa_r^2}{D^2}\quad(n=1,2,\ldots,\infty).$$

The functions $J_r^{(2)}(z,\tau)$ and $J_{(nr)}^{(2)}(z,\tau)$ are determined by expression (5-40), but in them we should assume $m_i = 0$, and also $K_{pr}\to\varkappa_r^2/D^2$ in the first case and $K_{pr}\to K_{nr}$ in the second case.

The transition from the spatial problem to the planar problem is obvious.

Let us discuss only the linear problem, since the temperature field of a significant portion of the filled seam and the adjacent area of the concrete column can be approximately considered homogeneous.

The problem is formulated as

$$\frac{\partial T}{\partial\tau}=a\frac{\partial^2T}{\partial x^2}+\frac{1}{c\gamma}q_0e^{-m\tau}\varepsilon(x)-H(T-T_2),$$

$$T(x,0)=F(x)=\begin{cases}f_0(x)\ \text{where}\ 0<x<L_1,\\ T_0\ \text{where}\ L_1<x<L;\end{cases}$$

$$\frac{\partial T(0,\tau)}{\partial x}=\frac{\partial T(L,\tau)}{\partial x}=0. \tag{5-44}$$

Here, as earlier

$$\varepsilon(x)=\begin{cases}1\ \text{where}\ L_1<x<L,\\ 0\ \text{where}\ 0<x<L_1.\end{cases}$$

The methods used in § 3-3 produce

$$
\begin{aligned}
T = T_2 + e^{-H\tau}\Bigg\{ & \frac{1}{L}\int_0^{L_1} f_\delta(x)\,dx + \left(1-\frac{L_1}{L}\right)T_0 - T_2 + \\
& + \frac{q_0}{c\gamma(m-H)}\left(1-\frac{L_1}{L}\right)\left(1-e^{-(m-H)\tau}\right) + \\
& + 2\sum_{n=1}^{\infty}\cos n\pi\frac{x}{L}\, e^{-\frac{n^2\pi^2}{L^2}a\tau}\left[\frac{1}{L}\int_0^{L_1} f_\delta(x)\cos n\pi\frac{x}{L}\,dx - \right. \\
& \left. - T_0\frac{\sin n\pi\frac{L_1}{L}}{n\pi} + \frac{q_0 L^2}{\lambda}\frac{\sin n\pi\frac{L_1}{L}}{n\pi(n^2\pi^2-m^{*2})}\right] - \\
& - 2\frac{q_0 L^2}{\lambda}e^{-(m-H)\tau}\sum_{n=1}^{\infty}\frac{\sin n\pi\frac{L_1}{L}}{n\pi(n^2\pi^2-m^{*2})}\cos n\pi\frac{x}{L}\Bigg\},
\end{aligned}
\tag{5-45}
$$

where

$$
m^{*2} = \frac{(m-H)L^2}{a}.
$$

In order to add up the last series in expression (5-45), we analyze the supplementary problem

$$
\begin{aligned}
& \frac{d^2u}{dx^2} + \frac{m^{*2}}{L^2}u = 1 - \varepsilon(x); \\
& \frac{du(0)}{dx} = \frac{du(L)}{dx} = 0.
\end{aligned}
\tag{5-46}
$$

Solving problem (5-46) by the method of finite integral transforms, we find:

$$
u = \frac{L^2}{m^{*2}}\frac{L_1}{L} - 2L^2\sum_{n=1}^{\infty}\frac{\sin n\pi\frac{L_1}{L}}{n\pi(n^2\pi^2-m^{*2})}\cos n\pi\frac{x}{L}.
$$

On the other hand, the solution of problem (5-46) in closed form is:

$$u=\frac{L^2}{m^{*2}}\left(\frac{\sin m^*\frac{L_1}{L}}{\sin m^*}-\frac{m^*}{L}\right)\cos m^*\frac{x}{L}+\frac{L}{m^*}$$

$$\text{при } 0<x<L_1;$$

$$u=\frac{L^2}{m^{*2}}\cdot\frac{\sin m^*\frac{L_1}{L}}{\sin m^*}\cos m^*\frac{x}{L}+2\frac{L}{m^*}\sin m^*\frac{L_1}{2L}\times$$

$$\times\sin\frac{m^*(2x-L_1)}{2L}\quad\text{при } L_1<x<L.$$

Thus,

$$-2L^2\sum_{n=1}^{\infty}\frac{\sin n\pi\frac{L_1}{L}}{n\pi(n^2\pi^2-m^{*2})}\cos n\pi\frac{x}{L}=$$

$$=\begin{cases}\dfrac{L^2}{m^{*2}}\left(\dfrac{\sin m^*\frac{L_1}{L}}{\sin m^*}-\dfrac{m^*}{L}\right)\cos m^*\dfrac{x}{L}+\dfrac{L}{m^*}\left(1-\dfrac{1}{m^*}\right)\\ \qquad (0<x<L_1);\\ \dfrac{L^2}{m^{*2}}\,\dfrac{\sin m^*\frac{L_1}{L}}{\sin m^*}\cos m^*\dfrac{x}{L}+2\dfrac{L}{m^*}\sin m^*\dfrac{L_1}{2L}\times\\ \times\sin\dfrac{m^*(2x-L_1)}{2L}-\dfrac{L^2}{m^{*2}}\,\dfrac{L_1}{L}\ (L_1<x<L).\end{cases}$$

As an example, Figure 5-10 presents results of the calculation of the temperature of a mass after concreting of a wide cooling seam. We assumed: width of seam concreted -- 1.5 m, width of block -- 15.0 m; a = 0.003 m^2/hr, λ = 1.7 kcal/(m·hr·C), $q_0/c\gamma$ = 0.4 C/hr, m = 0.01 1/hr.

The temperature graphs of Figure 5-10 are constructed with a displaced coordinate origin and with different scales on the abscissa for the seam and the block.

Calculation of Temperature Field of a Concrete Column Constructed in a Massive Block Deck

As A. M. Gindin [27] and K. V. Alekseyev [7] indicate, the use of a massive deck of concrete block 0.7 m high under the severe climatic conditions of construction of the Bratsk Power Plant has proven quite suitable. Heated in a tent to 5 C on the inside, it protected the concrete which had already been poured from rapid freezing and later from wide daily fluctuations in temperature, retaining a favorable temperature-humidity mode for the curing

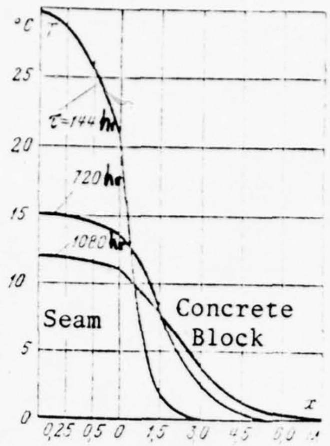

Figure 5-10. Temperature Field of Mass after Concreting of Broad Seam

concrete, etc. The positive experience of the Bratsk Power Plant indicates that a massive block deck could also be used in the construction of other hydraulic engineering structures in eastern Siberia.

As preliminary calculations have shown (see § 2-3), consideration of the thermal protective properties of a block deck by means of the effective heat transfer coefficient leads to inaccuracies in determination of the temperature field in the concrete.

This forces us to analyze the problem in a more precise statement.

Formulation of the problem. The system of differential equation

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T_0}{\partial z_{\bar{n}}^2} + \frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2}\right)$$

$$\left(\sum_{j=1}^{\bar{n}} R_j < z_{\bar{n}} < \infty,\ -L < x < L,\ -D < y < D,\ \tau > 0\right);$$

$$\frac{\partial T_s}{\partial \tau^{(\bar{n})}} = a\left(\frac{\partial^2 T_s}{\partial z_{\bar{n}}^2} + \frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2}\right) + \frac{1}{c\gamma} q_0 \varepsilon(x)\, \omega(y)\, e^{-m\tau},$$

$$\left(\sum_{j=s+1}^{\bar{n}} R_j < z_{\bar{n}} < \sum_{j=s}^{\bar{n}} R_j,\ -L<x<L,\ -D<y<D,\ \tau>0;\ s=1, 2, \ldots, \bar{n}\right) \quad (5\text{-}47)$$

The initial conditions

$$T_0(z_{\bar{n}}, x, y, 0) = T_0^{\text{cк}} + \varphi_0^{(\bar{n}-1)}(z_{\bar{n}}, x, y, \tau_{\bar{n}})$$

$$\left(\sum_{j=1}^{\bar{n}} R_j \leqslant z_{\bar{n}} < \infty, \ -L \leqslant x \leqslant L, \ -D \leqslant y \leqslant D\right);$$

$$T_s(z_{\bar{n}}, x, y, 0) = T_s + \varphi_s^{(\bar{n}-1)}(z_{\bar{n}}, x, y, \tau_{\bar{n}}),$$

$$\left(\sum_{j=s+1}^{\bar{n}} R_j \leqslant z_{\bar{n}} < \sum_{j=s}^{n} R_j, \ -L \leqslant x \leqslant L, \ -D \leqslant y \leqslant D; \ s = 1, 2, \ldots, n-1\right);$$

$$T_{\bar{n}}(z_{\bar{n}}, x, y, 0) = T^{(\bar{n})}\varepsilon(x)\omega(y) + T_{\text{оп}}[1 - \varepsilon(x)\omega(y)]$$

$$(0 \leqslant z_{\bar{n}} \leqslant R_{\bar{n}}, \ -L \leqslant x \leqslant L, \ -D \leqslant y \leqslant D).$$

The boundary conditions and conjugation conditions at "block-block" and "block-base" interfaces are the same as in (5-3)-(5-5).

Here $T_{\text{оп}}$ is the initial temperature of the deck;
$\varepsilon(x)$ and $\omega(y)$ are unit functions, equal to

$$\varepsilon(x) = \begin{cases} 1 & \text{where } -L_1 < x < L_1, \\ 0 & \text{where } -L < x < -L_1, \ L_1 < x < L; \end{cases}$$

$$\omega(y) = \begin{cases} 1 & \text{where } -D_1 < y < D_1, \\ 0 & \text{where } -D < y < -D_1, \ D_1 < y < D. \end{cases}$$

The solution of the problem

$$T = (T)_1 + (T)_2, \tag{5-48}$$

where $(T)_1$ is the temperature function, defined by the basic solutions of § 5-2 [formulas (5-6)], in which q_0 should be replaced by $q(\sin\mu_p \sin\kappa_r)^{-1}$ $\sin\mu_p\,(L_1/L)\sin\kappa_r\,(D_1/D)$; $(T)_2$ is a temperature function, equal to

$$(T)_2 = (T_{\text{оп}} - T^{(\bar{n})})\sum_{p=1}^{\infty}\sum_{r=1}^{\infty}\left(A_p - \frac{L\sin\mu_p\frac{L_1}{L}}{\mu_p\|V_0\|^2}\right)\times$$

$$\times\left(B_r - \frac{D\sin\varkappa_r\frac{L_1}{L}}{\varkappa_r\|V_0\|^2}\right)\cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D}\exp[-aK_{pr}\tau]\times$$

$$\times\left\{\operatorname{erfc}\left[\frac{z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}}\right] - \frac{1}{2}\operatorname{erfc}\left[\frac{R_{\bar{n}} - z_{\bar{n}}}{2\sqrt{a\tau^{(\bar{n})}}}\right] +\right.$$

$$+\frac{1}{2}\,\mathrm{erfc}\left[\frac{R_{\bar n}+z_{\bar n}}{2\sqrt{a\tau^{(\bar n)}}}\right]+\exp[h^2_{z(\bar n)}a\tau^{(\bar n)}+h_{z(\bar n)}z_{\bar n}]\times$$

$$\times\mathrm{erfc}\left[\frac{z_{\bar n}}{2\sqrt{a\tau^{(\bar n)}}}+h_{z(\bar n)}\sqrt{a\tau^{(\bar n)}}\right]-\exp[h^2_{z(\bar n)}a\tau^{(\bar n)}+$$

$$+h_{z(\bar n)}(R_{\bar n}+z_{\bar n})]\,\mathrm{erfc}\left[\frac{R_{\bar n}+z_{\bar n}}{2\sqrt{a\tau^{(\bar n)}}}+h_{z(\bar n)}\sqrt{a\tau^{(\bar n)}}\right]\Big\}. \qquad (5\text{-}49)$$

Regularization of the Temperature Field of Concrete Mass

Figure 5-11 presents the distribution of temperature through the height of a mass at the moment it is covered by the next block. The blocks are of equal height, 3 m, placed one upon the other at equal time intervals (60 and 240 hr)[1], the initial temperature of the concrete upon placement is 10 C, the initial temperature of the base is 4 C, the ambient temperature is -10 C; the heat-physical characteristics of the concrete in the base are: $a = 0.003\ m^3/hr$, $\lambda = 1.7$ kcal/(m·hr·C); the heat liberation parameters are: $q_0/c\gamma = 0.4$ C/hr, $m = 0.01$ 1/hr, the heat transfer function from the horizontal surface $\alpha = 20$ kcal/(m^2·hr·C).

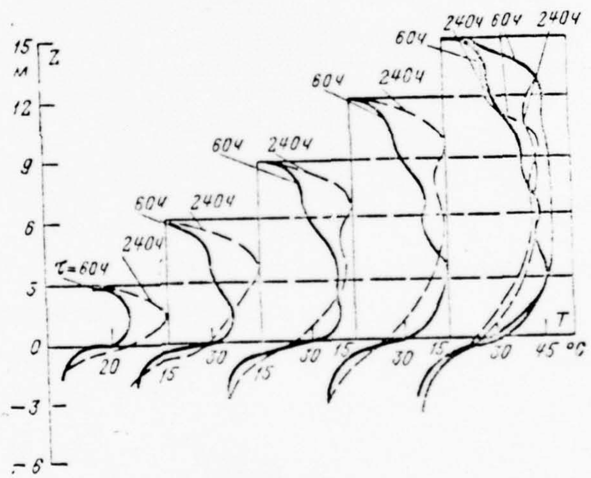

Figure 5-11. Temperature Field of Regularly Constructed Concrete Mass (One-Dimensional Problem)

[1]This growth rate of the mass in height is clearly excessive.

Our attention is drawn by the reguliarization of the temperature field of the masses erected in blocks of equal height with constant rate of growth of the structure as to height and constant initial and boundary conditions. This factor has been reported by many authors [15, 18]. The regularization of the temperature field was used in development of well known methods of calculation of the thermal mode of concrete masses during construction (see [22, 156, 172]).

Consideration of the Arbitrary Nature of the Dependence of Exothermy of Concrete on Time

All of the solutions of § 5-2 and 5-3 were produced on the assumption that the heat liberation intensity function is

$$q = q_0 e^{-m\tau}.$$

However, the dependence of exothermy of hydraulic engineering concretes on time is more complex in nature. However, as was indicated in § 2-2, it is satisfactorily described by the formula

$$q = \sum_{\nu=1}^{i} q_{0\nu} e^{-\nu m\tau}. \tag{5-50}$$

Under these conditions, the solution of the problems analyzed earlier can be simply extended to the case of any curve of heat liberation, approximated in accordance with expression (5-50), namely by assuming

$$T = (T)_1 + \sum_{\nu=2}^{i} (T)_\nu,$$

where $(T)_1$ is the solution of the problem with the heat liberation intensity function $q = q_{01}\exp[-m\tau]$ with fixed initial and boundary conditions; $(T)_{\nu(\nu=1,\ 2,\dots,\ i)}$ is the solution of the problem with intensity function $q_{0\nu}\exp[-\nu m\tau]$ with zero initial and boundary conditions.

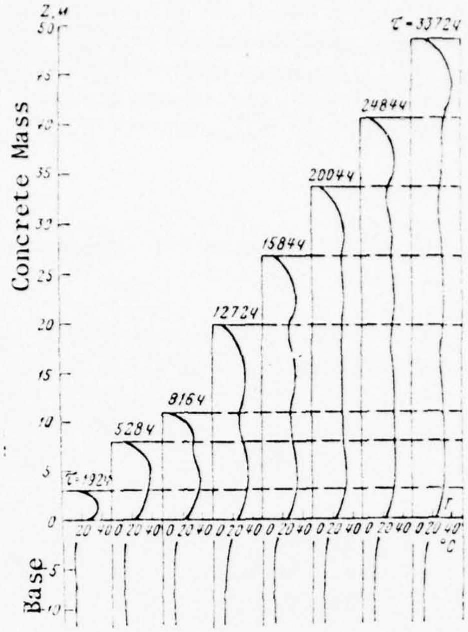

Figure 5-12. Temperature Field of a Concrete Mass in One Section of the Dam of the Bukhtarminskaya Power Plant

Temperature Field of the Concrete Mass in a Section of the Bukhtarminskaya Power Plant Dam

The basic chracteristics of the concrete blocks are presented in Table 5-9.

We assume: $a = 0.0025$ m^2/hr, $q_0/c\gamma = 0.411$ C/hr, $m = 0.0137$ 1/hr, $h = \alpha/\lambda =$ $= 13.3$ 1/m, the initial temperature of the rock is 11 C.

The calculation was performed to determine the temperature through the axis of the mass, the temperature field was looked upon as homogeneous. The results of calculations are presented graphically in Figure 5-12.

TABLE 5-9. BASIC CHARACTERISTICS OF CONCRETE BLOCKS

№ блока	Высота блока, м	Время установки блока, ч	Температура бетонной смеси при укладке блока, °С	Температура среды на горизонтальной поверхности блока, °С
1	3	0	11	11
2	5	192	9	9
3	3	528	4	4
4	9	816	2	5
5	7	1 272	2	5
6	7	1 584	2	5
7	7	2 304	4	4
8	8	3 184	12	12

Key: 1, Block Number; 2, Block Height, m; 3, Block Age, hr; 4, Temperature of Concrete Mixture upon Placement of Block, C; 5, Ambient Temperature of Horizontal Surface of Block, C

5-4. Finite Difference Methods of Calculation of Temperature Fields in Concrete Masses Growing Block by Block

One-Dimensional Temperature Field

Let us place the coordinate origin at a certain distance R_0 from the boundary separating the concrete mass and the base, and direct the OZ axis upward. Quantity R_0 is selected so that during the course of the process, the thermal perturbation developing in the base does not reach this point. Suppose $0 < z < R_0$ is the calculated area of the base, $R_0 < z < \sum_{j=1}^{\bar{n}} R_j$ is the calculated area of the concrete mass. Then the problem is formulated as follows:

The system of differential equations

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a_0 \frac{\partial^2 T_0}{\partial z^2} - H_0 (T_0 - T^{\text{ск}})$$

$$(0 < z < R_0, \ \tau > 0);$$

$$\frac{\partial T_s}{\partial \tau^{(\bar{n})}} = a_s \frac{\partial^2 T_s}{\partial z^2} - H_s (T_s - \gamma_s) + \frac{1}{c\gamma} q_v^{(s)} (d_v^{(s)} + b_v^s T_s) \exp[-m_v^{(s)} t_s]$$

$$\left(\sum_{j=s-1}^{s} R_j < z < \sum_{j=s-1}^{s+1} R_j, \ \tau > 0; \ s = 1, 2, \ldots, \bar{n} \right); \qquad (5\text{-}51)$$

The initial conditions

$$T_0(z,\ 0)=f_0(z);$$

$$T_s(z,\ 0)=f_s(z);$$

(5-52)

The boundary conditions at the upper horizontal surface

$$T_{\bar{n}}\left(\sum_{j=0}^{\bar{n}} R_j,\ \tau^{(\bar{n})}\right)=\varphi_{\bar{n}}(\tau^{(\bar{n})})$$

(boundary conditions of first kind), or

$$\frac{\partial T_{\bar{n}}\left(\sum_{j=0}^{\bar{n}} R_j,\ \tau^{(\bar{n})}\right)}{\partial z}=-h_{z(\bar{n})}\left[\varphi_{\bar{n}}(\tau^{(\bar{n})})-T_{\bar{n}}\left(\sum_{j=0}^{\bar{n}} R_j,\ \tau^{(\bar{n})}\right)\right] \tag{5-53}$$

(boundary conditions of third kind);

At the lower surface

$$\frac{\partial T_0(0,\ \tau^{(\bar{n})})}{\partial z}=\text{const}$$

or

$$T_0(0,\ \tau^{(\bar{n})})=\text{const}.$$

At the division boundaries of the blocks, as well as the boundary between the bottom block and the base, the following conjugation conditions are assigned:

$$T_s\left(\sum_{j=0}^{s-1} R_j,\ \tau^{(\overline{n})}\right)=T_{s-1}\left(\sum_{j=0}^{s-1} R_j,\ \tau^{(\overline{n})}\right);$$

$$\lambda_s \frac{\partial T_s\left(\sum_{j=0}^{s-1} R_j,\ \tau^{(\overline{n})}\right)}{\partial z}=\lambda_{s-1} \frac{\partial T_{s-1}\left(\sum_{j=0}^{s-1} R_j,\ \tau^{(\overline{n})}\right)}{\partial z}. \tag{5-54}$$

As we can see from (5-51)-(5-54), the statement of the problem is quite general in nature. In addition to those factors which are considered in the corresponding analytic solutions, here each block, as well as the base, can be characterized by its own values of heat-physical characteristics a_s, λ_s, $c_s\gamma_s$, its own parameter H_s, the heat liberation intensity function is represented by a general algorithm allowing description of various forms of the dependence of exothermy on temperature and time, the parameters of this function $q_\nu^{(s)}$, $d_\nu^{(s)}$, $b_\nu^{(s)}$ and $m_\nu^{(s)}$ may vary from block to block, etc.

Basically all the symbols are similar to those used in the preceding sections: $\overline{n}$ is the number of blocks in a mass during an intermediate stage of construction ($\overline{n} = 1, 2, \ldots, n$); R_s is the height of the sth block ($s = 1, 2, \ldots, \overline{n}$); $\tau^{(\overline{n})}$ is the time from the moment of placement of the $\overline{n}$th block, t_s is the absolute time of "life" of the sth block; $q_\nu^{(s)}$, $d_\nu^{(s)}$, $b_\nu^{(s)}$ and $m_\nu^{(s)}$ are parameters of the heat liberation intensity function; $h_{z(\overline{n})}$ is the relative heat transfer coefficient; $\psi_{\overline{n}}(\tau^{(\overline{n})})$ is the ambient temperature at the horizontal surface (with boundary conditions of the third kind); $\phi_{\overline{n}}(\tau^{(\overline{n})})$ is the temperature of the horizontal surface (with boundary conditions of the first kind); $\chi_{\overline{n}}(\tau^{(\overline{n})})$ is the ambient temperature in the area of the concrete blocks at surfaces not included in the boundary conditions of the problem; $T^{CK}(\tau^{(\overline{n})})$ is a parameter similar to $\chi_{\overline{n}}$, but in the area of the base.

In order to approximate the differential equations (5-51), we can use an explicit four-point plan (see § 3-6).

Suppose $\ell_{\overline{n}}$ is constant, in the interval from the moment of placement of the $\overline{n}$th block until it is covered by the $(\overline{n} + 1)$th block, the time step ($\tau^{(\overline{n})} = k\ell_{\overline{n}}$, $k = 1, 2, \ldots$); h_s is the step in the coordinate, constant within the limits of each block, but such that for all blocks in the mass the number of divisions along the coordinate $i_{\overline{n}}$ is identical, while for the base is it m times greater (so that the coordinate, relative within the limits of the sth block, $z = ih_s$, $i = 1, 2, \ldots, i_{\overline{n}}$).

Thus, within the grid selected, the time steps and coordinate steps are variable, or more precisely piecewise-constant.

In order to approximate the nondifferential terms in the system of equations (5-51), we utilize the moment in time (k + 1/2), i.e., we assume

$$T \rightarrow \frac{T_{i,k} + T_{i,k+1}}{2}.$$

Then the basic calculation formulas become

$$T^{(s)}_{i,k+1} = A_s T^{(s)}_{i,k} + B_s (T^{(s)}_{i+1,k} + T^{(s)}_{i-1,k}) + C_s \quad (s=0,\ 1,\ 2, \ldots,\ \bar{n}), \tag{5-55}$$

where for internal points of the concrete mass (s = 1, 2,..., $\bar{n}$)

$$A_s = \frac{1}{D_s}\left(1 - 2M_s + \frac{1}{2}\frac{q_v^{(s)} b_v^{(s)}}{c_s\gamma_s} l_{\bar{n}} \exp[-m_v^{(s)} \tau^{(\bar{n})}_{k+1/2}] - \frac{1}{2} H_s l_{\bar{n}}\right),$$

$$B_s = \frac{M_s}{D_s};\quad M_s = \frac{a_s l_{\bar{n}}}{h_s^2};$$

$$C_s = \frac{1}{D_s}\left(\frac{q_v^{(s)} d_v^{(s)}}{c_s\gamma_s} \exp[-m_v^{(s)} \tau^{(\bar{n})}_{k+1/2}] + H_s \gamma_s l_{\bar{n}}\right);$$

$$D_s = 1 - \frac{1}{2}\frac{q_v^{(s)} b_v^{(s)}}{c_s\gamma_s} l_{\bar{n}} \exp[-m_v^{(s)} \tau^{(\bar{n})}_{k+1/2}] + \frac{1}{2} H_s l_{\bar{n}};$$

for internal points in the base (s = 0)

$$A_0 = \frac{1}{D_0}\left(1 - 2M_0 - \frac{1}{2} H_0 l_{\bar{n}}\right);$$

$$B_0 = \frac{M_0}{D_0};\quad M_0 = \frac{a_0 l_{\bar{n}}}{h_0^2};$$

$$C_0 = \frac{1}{D_0} H_0 T^{\text{cK}} l_{\bar{n}};$$

$$D_0 = 1 + \frac{1}{2} H_0 l_{\bar{n}}.$$

The condition of stability of these formulas can be produced by the method outlined in § 3-6.

For our purposes, it is sufficient to limit ourselves to the equation

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial z^2} + \frac{qb}{c\gamma}e^{-m\tau}T - HT.$$

Its difference analog

$$\frac{T_{i,k+1} - T_{i,k}}{l} = a\Lambda T^{(k)} + \frac{qb}{c\gamma}e^{-m\tau_{k+1/2}}\frac{T_{i,k+1} + T_{i,k}}{2} - H\frac{T_{i,k+1} + T_{i,k}}{2},$$

where

$$\Lambda T^{(k)} = \frac{1}{h^2}(T_{i+1,k} - 2T_{i,k} + T_{i-1,k}),$$

ℓ and h are the time and coordinate steps ($\tau = k\ell$, $z = ih$; $k = 1, 2, \ldots$; $i = 1, 2, \ldots, i_n$).

Let us separate the variables x and τ, assuming

$$T = Z(z)\theta(\tau).$$

We have

$$Z\frac{\theta_{k+1} - \theta_k}{l} = a\theta_k\Lambda Z + \frac{1}{2}\frac{qb}{c\gamma}e^{-m\tau_{k+1/2}}Z\theta_{k+1} +$$
$$+ \frac{1}{2}\frac{qb}{c\gamma}e^{-m\tau_{k+1/2}}Z\theta_k - \frac{1}{2}HZ\theta_{k+1} - \frac{1}{2}HZ\theta_k.$$

From this it follows that

$$\frac{\theta_{k+1} - \theta_k}{al\theta_k} - \frac{1}{2}\frac{qb}{ac\gamma}e^{-m\tau_{k+1/2}}\frac{\theta_{k+1}}{\theta_k} - \frac{1}{2}\frac{qb}{ac\gamma}e^{-m\tau_{k+1/2}} +$$
$$+ \frac{1}{2}\frac{H}{a}\frac{\theta_{k+1}}{\theta_k} + \frac{1}{2}\frac{H}{a} = \frac{\Lambda Z}{Z} = -\mu$$

or

$$\theta_{k+1} = p_k \theta_k,$$

where μ is the Eigenvalue of the Shturm-Liouville difference problem, as in § 3-6, equal to

$$\mu_\nu = \frac{4}{h^2} \sin^2 \frac{\nu \pi h}{R} \quad (\nu = 1, 2, \ldots, i_{\bar{n}} - 1);$$

p_k is the conversion factor, equal to

$$p_k = \left(1 - \frac{1}{2} \frac{qb}{c\gamma} l e^{-m\tau_{k+1/2}} + \frac{1}{2} Hl\right)^{-1} \times$$
$$\times \left(1 + \frac{1}{2} \frac{qb}{c\gamma} l e^{-m\tau_{k+1/2}} - \frac{1}{2} Hl - \mu a l\right).$$

As it is not difficult to see, the stability condition

$$|p_k| \leqslant 1$$

or

$$-1 \leqslant p_k \leqslant 1$$

is satisfied if the time step l is subject to the limitations

$$l \leqslant \frac{h^2}{2a}.$$

In order to approximate the boundary conditions of the third and second kind and the conjugation conditions, we can utilize finite-difference approximations produced as a result of application of the method of thermal balances to the elements of the surface of the mass, the "block-block" and "block-base" separation interfaces.

The thermal balance equation for the shaded element of the volume near boundary Γ (Figure 5-3) in the case of convective heat exchange between the body and the environment (boundary condition of the third kind) is

$$c\gamma \frac{h}{2} \frac{T_{\Gamma,k+1} - T_{\Gamma,k}}{l} = \alpha(\psi - T_{\Gamma,k}) + \lambda \frac{T_{\Gamma-1,k} - T_{\Gamma,k}}{h} +$$
$$+ \frac{h}{2} q_\nu \left(d_\nu + b_\nu \frac{T_{\Gamma,k+1} + T_{\Gamma,k}}{2}\right) e^{-m_\nu \tau_{k+1/2}} -$$
$$- \frac{h}{2} Hc\gamma \left(\frac{T_{\Gamma,k+1} + T_{\Gamma,k}}{2} - \chi\right).$$

Here ψ is the ambient temperature; α is the heat transfer coefficient; h is the step along the coordinate.

It follows from this that:

$$T^{(\overline{n})}_{\Gamma,k+1} = A_{\overline{n}} T^{(\overline{n})}_{\Gamma,k} + B_{\overline{n}} T^{(\overline{n})}_{\Gamma+1,k} + C_{\overline{n}}, \tag{5-56}$$

where

$$A_{\overline{n}} = \frac{1}{D_{\overline{n}}} \left(1 - 2M_{\overline{n}} - 2N_{\overline{n}} M_{\overline{n}} + \frac{1}{2} \frac{q_\nu^{(\overline{n})} b_\nu^{(\overline{n})}}{c_{\overline{n}} \gamma_{\overline{n}}} \times\right.$$
$$\left. \times l_{\overline{n}} \exp\left[-m_\nu^{(\overline{n})} \tau^{(\overline{n})}_{k+1/2}\right] - \frac{1}{2} H_{\overline{n}} l_{\overline{n}} \right);$$

$$B_{\overline{n}} = 2 \frac{M_{\overline{n}}}{D_{\overline{n}}}; \quad M_{\overline{n}} = \frac{a_{\overline{n}} l_{\overline{n}}}{h^2_{\overline{n}}}; \quad N_{\overline{n}} = \frac{\alpha_{\overline{n}} l_{\overline{n}}}{\lambda_{\overline{n}}};$$

$$C_{\overline{n}} = \frac{1}{D_{\overline{n}}} \left(2M_{\overline{n}} N_{\overline{n}} \psi_{\overline{n}} (\tau_{k+1/2}) + \frac{q_\nu^{(\overline{n})} b_\nu^{(\overline{n})}}{c_{\overline{n}} \gamma_{\overline{n}}} \times\right.$$
$$\left. \times l_{\overline{n}} \exp\left[-m_\nu^{(\overline{n})} \tau^{(\overline{n})}_{k+1/2}\right] + H_{\overline{n}} \chi_{\overline{n}} l_{\overline{n}} \right); \tag{5-57}$$

$$D_{\overline{n}} = 1 - \frac{1}{2} \frac{q_\nu^{(\overline{n})} b_\nu^{(\overline{n})}}{c_{\overline{n}} \gamma_{\overline{n}}} l_{\overline{n}} \exp\left[-m_\nu^{(\overline{n})} \tau^{(\overline{n})}_{k+1/2}\right] + \frac{1}{2} H_{\overline{n}} l_{\overline{n}}. \tag{5-58}$$

If the heat flux η(τ) is fixed on the surface (boundary condition of the second kind), which, for example, is the case on the lower boundary of the base [see (5-53)], in this case the structure of the calculation formulas does not change, but we must assume:

$\overline{n} = 0$, $N_{\overline{n}} = 0$, $q_\nu = 0$, in formula (5-57) we replace

$H_{\bar{n}} \chi_{\bar{n}} \ell_{\bar{n}} \rightarrow H_0 T^{CK} \ell_{\bar{n}}$, and in formula (5-58) we add the term

$$h/\lambda \times \eta(\tau_{k+1/2}).$$

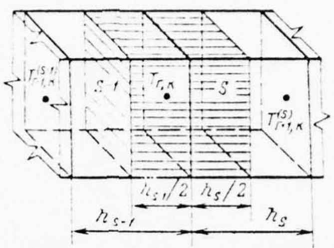

Figure 5-13. Elementary Sector on Line Between Blocks

In the case of a boundary condition of the second kind such as

$$\left.\frac{\partial T}{\partial x}\right|_{x=\Gamma}=0, \tag{5-59}$$

usually used as a symmetry condition, the temperature at the boundary $T_{\Gamma,k+1}$ is determined from formula (5-56), if we assume in it $N_{\bar{n}} = 0$.

Figure 5-13 presents an elementary sector on the line of separation of blocks. It is assumed that the thermal contact between blocks is perfect, i.e., $T_{\Gamma+0} = T_{\Gamma-0}$.

The equation for the thermal balance for an elementary volume encompassing a sector of the mating line leads to the algorithm

$$T_{\Gamma,k+1}=AT_{\Gamma,k}+G_sT_{\Gamma+1,k}+G_{s-1}T_{\Gamma-1,k}+C, \tag{5-60}$$

where

$$A=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_jc_j\gamma_j\left(1-2M_j+\frac{1}{2}\frac{q_v^{(j)}b_v^{(j)}}{c_j\gamma_j}l_{(\bar{n})}\times\right.$$
$$\left.\times\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\bar{n})}\right]-\frac{1}{2}H_jl_{\bar{n}}\right);$$
$$G_p=\frac{1}{D}\frac{\lambda_pl_{\bar{n}}}{h_p}\quad(p=s,\ s-1);\ M_j=\frac{a_jl_{\bar{n}}}{h_j^2};$$

$$C=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_j c_j \gamma_j\left(\frac{q_v^{(j)}d_v^{(j)}}{c_j\gamma_j}l_{\bar{n}}\exp\left[-m_v^{(j)}\tau_{k+\frac{1}{2}}^{(\bar{n})}\right]+H_j\gamma_j l_{\bar{n}}\right);$$

$$D=\sum_{j=s-1}^{s}\frac{1}{2}h_j c_j \gamma_j\left(1-\frac{1}{2}\frac{q_v^{(j)}b_v^{(j)}}{c_j\gamma_j}l_{\bar{n}}\exp\left[-m_v^{(j)}\tau_{k+\frac{1}{2}}^{(\bar{n})}\right]+\right.$$

$$\left.+\frac{1}{2}H_j l_{\bar{n}}\right).$$

The calculation formulas produced can be analyzed for stability by using a method suggested by P. Price and M. Slack [164].

Suppose for certain $\tau = k\ell$ the maximum error for the value of $T_{i,k}$ with variable i and constant k is ε_k.

Then for stability of the calculation plan it is sufficient to require that

$$\varepsilon_{k+1}\leqslant\varepsilon_k. \tag{5-61}$$

Obviously

$$\varepsilon_k=\max|\Delta T_{i,k}|.$$

From formula (5-56) we find

$$|\Delta T_{r,k+1}|\leqslant|A_{\bar{n}}|\,|\Delta T_{r,k}|+|B_{\bar{n}}|\,|\Delta T_{r-1,k}|.$$

From this

$$\varepsilon_{k+1}\leqslant|A_{\bar{n}}|\,\varepsilon_k+|B_{\bar{n}}|\,\varepsilon_k.$$

But

$$|B_{\bar{n}}|=2M_{\bar{n}}\left|\frac{1}{D_{\bar{n}}}\right|.$$

and if we assume that $D_{\bar{n}} > 0$[1], then

$$|B_{\bar{n}}| = \frac{2M_{\bar{n}}}{D_{\bar{n}}}.$$

Consequently

$$\varepsilon_{k+1} \leqslant \{|A_{\bar{n}}| + B_{\bar{n}}\} \varepsilon_k. \quad (5\text{-}62)$$

Comparison of expressions (5-61) and (5-62) yields

$$|A_{\bar{n}}| + B_{\bar{n}} \leqslant 1.$$

Thus, the stability condition with respect to the boundary conditions of the third kind is

$$M_{\bar{n}}(2 + N_{\bar{n}}) \leqslant 1$$

or

$$l \leqslant \frac{h^2_{\bar{n}}}{a_{\bar{n}}(2 + N_{\bar{n}})}. \quad (5\text{-}63)$$

The stability condition with respect to boundary conditions of the second kind at the lower boundary of the base

$$l \leqslant \frac{h_0^2}{2a_0}. \quad (5\text{-}64)$$

The stability condition for grid function (5-60), yielding the temperature on the "block-block" and "block-base" interfaces, agrees with (5-59), where we must analyze the lower value of the ratio h^2/a.

---

[1] For concrete masses considering those values of $l_{\bar{n}}$ produced under conditions of stability, this assumption is always fulfilled.

Thus, in calculation of a homogeneous temperature field of the mass of $\bar{n}$ blocks using an explicit four-point finite-difference system, the time step $\ell_{\bar{n}}$ should satisfy the conditions

$$l_{\bar{n}} \leqslant \frac{h_s^2}{2a_s} \quad (s = 0,\ 1, \ldots,\ \bar{n});$$

$$l_{\bar{n}} \leqslant \frac{h_{\bar{n}}^2}{a_{\bar{n}}(2 + N_{\bar{n}})}.$$

Two-Dimensional (Planar) Temperature Field

A study is made of the temperature field in the process of block-by-block growth of a concrete column located on a base consisting of a half plane.

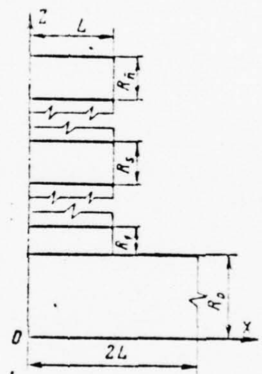

Figure 5-14. Plan of Calculation Area in Problem of Planar Temperature Field of Mass

The geometry of the calculation area and the system of coordinates are presented in Figure 5-14. The base is replaced with a rectangle, the width of which is equal to twice the width of the block. Depth R_0 is selected from the same considerations as in the one-dimensional problem.

The system of differential equations

$$\frac{\partial T_0}{\partial \tau^{(\bar{n})}} = a_0 \left(\frac{\partial^2 T_0}{\partial z^2} + \frac{\partial^2 T_0}{\partial x^2} \right) - H_0 (T_0 - T^{\text{cK}})$$

$$(0 < z < R_0, \ -2L < x < 2L, \ \tau > 0).$$

$$\frac{\partial T_s}{\partial \tau^{(\bar{n})}} = a_s \left(\frac{\partial^2 T_s}{\partial z^2} + \frac{\partial^2 T_s}{\partial x^2} \right) - H_s (T_s - \chi_s) +$$

$$+ \frac{1}{c_s \gamma_s} q_*^{(s)} (d_*^{(s)} + b_*^{(s)} T_s) \exp[-m_*^{(s)} t_s]$$

$$\left(\sum_{j=s-1}^{s} R_j < z < \sum_{j=s-1}^{s+1} R_j, \ -L < x < L, \ \tau > 0; \ s = 1, 2, \ldots, \bar{n} \right). \quad (5\text{-}65)$$

The initial conditions

$$T_0 (z, x, 0) = f_0 (z, x);$$

$$T_s (z, x, 0) = f_s (z, x). \quad (5\text{-}66)$$

On the line $(x = 0, \ 0 < z < \sum_{j=0}^{\bar{n}} R_j)$ the condition of symmetry is assumed

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0. \quad (5\text{-}67)$$

Condition (5-67) is also used on the right boundary of the base ($x = 2L, \ 0 < z < R_0$).

At the lower boundary of the base ($z = 0, \ -2L < x < 2L$) it is assumed that

$$\left. \frac{\partial T}{\partial z} \right|_{z=0} = \text{const}$$

or

$$T|_{z=0} = \text{const}.$$

At the boundaries of the concrete mass $\left(z=\sum_{i=0}^{\bar{n}} R_i;\ -L<x<L\right)$ and $\left(x=L,\ R_0<z<\sum_{i=0}^{\bar{n}} R_i\right)$, and also at the boundary of the base ($z = R_0$, $-2L < x < -L$, $L < x < 2L$) boundary conditions of either the first or the third kind are assigned.

In the latter case, any combination of types of boundary conditions on the contour of the body is possible, as is a change in boundary conditions with time. In the general case, various ambient temperatures or surface temperatures are studied on the various surfaces of the mass.

Differential equations (5-65) are approximated using an explicit six point plan of finite difference relationships in a rectangular grid.

Suppose, as before, $l_{\bar{n}}$ is the time step, h_z and h_x are the steps on coordinates OZ and OX ($\tau = kl_{\bar{n}}$, $z = ih_z$, $x = jh_x$).

Then the grid functions for the sth block become

$$T^{(s)}_{i,j,k+1}=A_sT^{(s)}_{i,j,k}+B^{(s)}_z(T^{(s)}_{i+1,j,k}+T^{(s)}_{i-1,j,k})+ \\ +B^{(s)}_x(T^{(s)}_{i,j+1,k}+T^{(s)}_{i,j-1,k})+C_s\ (s=0,\ 1,\ 2,\ldots,\ \bar{n}), \quad (5\text{-}68)$$

where for the internal points of the concrete blocks ($s = 1, 2,\ldots, \bar{n}$):

$$A_s=\frac{1}{D_s}\left(1-2M^{(s)}_z-2M^{(s)}_x+\frac{1}{2}\frac{q^{(s)}_v b^{(s)}_v}{c_s\gamma_s}l_{\bar{n}}\times\right. \\ \left.\times\exp\left[-m^{(s)}_v\tau^{(\bar{n})}_{k+1/2}\right]-\frac{1}{2}H_sl_{\bar{n}}\right];$$

$$B^{(s)}_z=\frac{M^{(s)}_z}{D_s};\quad M^{(s)}_z=\frac{a_sl_{\bar{n}}}{(h^{(s)}_z)^2};$$

$$B^{(s)}_x=\frac{M^{(s)}_x}{D_s};\quad M^{(s)}_x=\frac{a_sl_{\bar{n}}}{(h^{(s)}_x)^2};$$

$$C_s=\frac{1}{D_s}\left(\frac{q^{(s)}_v d^{(s)}_v}{c_s\gamma_s}l_{\bar{n}}\exp\left[-m^{(s)}_v\tau^{(\bar{n})}_{k+1/2}\right]+H_s\gamma_s l_{\bar{n}}\right);$$

$$D_s=1-\frac{1}{2}\frac{q^{(s)}_v b^{(s)}_v}{c_s\gamma_s}l_{\bar{n}}\exp\left[-m^{(s)}_v\tau^{(\bar{n})}_{k+1/2}\right]+\frac{1}{2}H_sl_{\bar{n}};$$

for the internal points of the base ($s = 0$):

$$A_0 = \frac{1}{D_0}\left(1 - 2M_z^{(0)} - 2M_x^{(0)} - \frac{1}{2} H_0 l_{\bar{n}}\right);$$

$$B_z = \frac{M_z^{(0)}}{D_0}; \quad M_z^{(0)} = \frac{a_0 l_{\bar{n}}}{(h_z^{(0)})^2};$$

$$B_x = \frac{M_x^{(0)}}{D_0}; \quad M_x^{(0)} = \frac{a_0 l_{\bar{n}}}{(h_x^{(0)})^2};$$

$$C_0 = \frac{1}{D_0} H_0 T^{\text{ек}} l_{\bar{n}};$$

$$D_0 = 1 + \frac{1}{2} H_0 l_{\bar{n}}.$$

The depth of the base is not limited, but the number of divisions of depth is assumed equal to the number of divisions of width, and the junctions of the calculation area of the bottom block and base are matched.

Special calculations have shown that with a base width equal to twice the width of the concrete mass, sufficient accuracy of calculation of the temperature field in the concrete mass and in adjacent areas of the base is provided. Due to the need to expand the calculation area of the base, the coefficients should be altered.

For all internal points in the area, where $T \leq 0$ and, consequently, there is no heat liberation ($q_\nu = 0$), the corresponding coefficient are converted.

The stability condition

$$l_s \leqslant \frac{1}{2a_s\left[\frac{1}{(h_z^{(s)})^2} + \frac{1}{(h_x^{(s)})^2}\right]} \quad (s = 0, 1, \ldots, \bar{n}). \tag{5-69}$$

As for one-dimensional problems, the boundary conditions and interface conditions are approximated by algorithms which follow from the thermal balance equations for elementary sectors near the boundary of the body or interface line (Figure 5-15, shaded area).

These algorithms and the corresponding conditions of stability are:

1. The boundary of the body parallel to the OZ axis

$$T_{i,j,k+1}^{(s)} = A_z^{(s)} T_{i,j,k}^{(s)} + B_z^{(s)} (T_{i-1,j,k}^{(s)} + T_{i+1,j,k}^{(s)}) + 2B_x^{(s)} T_{i,j\pm 1,k}^{(s)} + C_z^{(s)}, \tag{5-70}$$

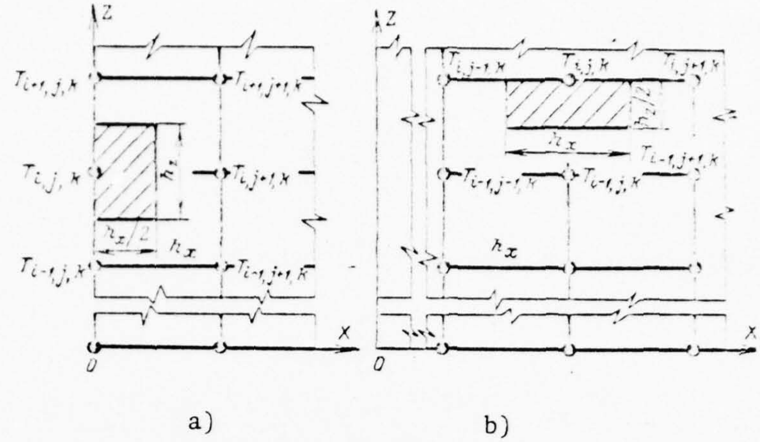

Figure 5-15. Elementary Sector Near Boundary of Body. a, Boundary of Body Parallel to OZ; b, Boundary of Body Parallel to OX

where

$$A_z^{(s)} = \frac{1}{D_s}\left(1 - 2M_z^{(s)} - 2M_x^{(s)} - 2N_x^{(s)} M_x^{(s)} + \right.$$

$$\left. + \frac{1}{2}\frac{q_v^{(s)} b_v^{(s)}}{c_s \gamma_s} l_{\bar{n}} \exp\left[-m_v^{(s)} \tau_{k+1/2}^{(\bar{n})}\right] - \frac{1}{2} H_s l_{\bar{n}}\right);$$

$$B_z^{(s)} = \frac{M_z^{(s)}}{D_s}; \quad M_z^{(s)} = \frac{a_s l_{\bar{n}}}{(h_z^{(s)})^2};$$

$$B_x^{(s)} = \frac{M_x^{(s)}}{D_s}; \quad M_x^{(s)} = \frac{a_s l_{\bar{n}}}{(h_x^{(s)})^2};$$

$$C_z^{(s)} = \frac{1}{D_s}\left(2N_x^{(s)} M_x^{(s)} \psi_{s(z)} + \frac{q_v^{(s)} d_v^{(s)}}{c_s \gamma_s} l_{\bar{n}} \exp\left[-m_v^{(s)} \tau_{k+1/2}^{(\bar{n})}\right] + \right.$$

$$\left. + H_s \chi_s l_{\bar{n}}\right);$$

$$N_x = \frac{\alpha_{s(z)} h_x^{(s)}}{\lambda_s};$$

$$D_s = 1 - \frac{1}{2}\frac{q_v^{(s)} b_v^{(s)}}{c_s \gamma_s} l_{\bar{n}} \exp\left[-m_v^{(s)} \tau_{k+1/2}^{(\bar{n})}\right] + \frac{1}{2} H_s l_{\bar{n}};$$

$\alpha_{s(z)}$ and $\psi_{s(z)}$ are the heat transfer coefficient and ambient temperature at the surface of the body parallel to OZ.

The stability condition

$$l_{\bar{s}} \leqslant \frac{1}{a_s\left[\frac{2}{(h_z^{(s)})^2} + \frac{1}{(h_x^{(s)})^2}(2 + N_x^{(s)})\right]}. \tag{5-71}$$

2. The boundary of the body parallel to the OX axis

$$\begin{gathered} T^{(\bar{n})}_{i,j,k+1} = A^{(\bar{n})}_x T^{(\bar{n})}_{i,j,k} + B^{(\bar{n})}_x (T^{(\bar{n})}_{i,j-1,k} + T^{(\bar{n})}_{i,j+1,k}) + \\ + 2B^{(\bar{n})}_z T^{(\bar{n})}_{i\pm 1,j,k} + C^{(n)}_x, \end{gathered} \tag{5-72}$$

where

$$A^{(\bar{n})}_x = \frac{1}{D_s}\Big(1 - 2M^{(\bar{n})}_x - 2M^{(\bar{n})}_z - 2N^{(\bar{n})}_z M^{(\bar{n})}_z +$$

$$+ \frac{1}{2}\frac{q_v^{(\bar{n})} b_v^{(\bar{n})}}{c_{\bar{n}}\gamma_{\bar{n}}} l_{\bar{n}} \exp\left[-m_v^{(\bar{n})}\tau^{(\bar{n})}_{k+1/2}\right] - \frac{1}{2} H_{\bar{n}} l_{\bar{n}}\Big);$$

$$C^{(\bar{n})}_z = \frac{1}{D_{\bar{n}}}\Big(2N^{(\bar{n})}_z M^{(\bar{n})}_z \psi_{s(x)} + \frac{q_v^{(\bar{n})} d_v^{(\bar{n})}}{c_{\bar{n}}\gamma_{\bar{n}}} l_{\bar{n}} \times$$

$$\times \exp\left[-m_v^{(\bar{n})}\tau^{(\bar{n})}_{k+1/2}\right] + H_{\bar{n}}\chi_{\bar{n}} l_{\bar{n}}\Big);$$

$$N^{(\bar{n})}_z = \frac{\alpha_{\bar{n}(x)} h_z}{\lambda_{\bar{n}}}.$$

The remaining symbols are the same as before.

The stability condition

$$l_{\bar{n}} \leqslant \frac{1}{a_{\bar{n}}\left[\frac{2}{(h_x^{(\bar{n})})^2} + \frac{1}{(h_z^{(\bar{n})})^2}(2 + N_z^{(\bar{n})})\right]}. \tag{5-73}$$

3. The interface lines parallel to the OX axis

$$\begin{gathered} T_{i,j,k+1} = AT_{i,j,k} + G_s T_{i+1,j,k} + G_{s-1} T_{i-1,j,k} + \\ + E(T_{i,j+1,k} + T_{i,j-1,k}) + C, \end{gathered} \tag{5-74}$$

where

$$A=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_z^{(j)}c_j\gamma_j\Big(1-2M_z^{(j)}-2M_x^{(j)}+$$

$$+\frac{1}{2}\frac{q_v^{(j)}b_v^{(j)}}{c_j\gamma_j}l_{\overline{n}}\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\overline{n})}\right]-\frac{1}{2}H_jl_{\overline{n}}\Big);$$

$$G_p=\frac{1}{D}\frac{\lambda_pl_{\overline{n}}}{h_z^p}\ (p=s,\ s-1);$$

$$M_z^{(j)}=\frac{a_jl_{\overline{n}}}{(h_z^{(j)})^2};\quad M_x^{(j)}=\frac{a_jl_{\overline{n}}}{(h_x^{(j)})^2};$$

$$E=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}\frac{h_z^{(j)}}{h_x^{(j)}}\frac{\lambda_jl_{\overline{n}}}{h_x^{(j)}};$$

$$C=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_z^{(j)}c_j\gamma_j\left(\frac{q_v^{(j)}d_v^{(j)}}{c_j\gamma_j}l_{\overline{n}}\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\overline{n})}\right]+H_j\chi_jl_{\overline{n}}\right);$$

$$D=\sum_{j=s-1}^{s}\frac{1}{2}h_z^{(j)}c_j\gamma_j\Big(1-\frac{1}{2}\frac{q_v^{(j)}b_v^{(j)}}{c_j\gamma_j}l_{\overline{n}}\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\overline{n})}\right]+$$

$$+\frac{1}{2}H_jl_{\overline{n}}\Big).$$

4. A point at the intersection of the boundary of section 1 and the interface line of section 3

$$T_{i,j,k+1}=A'T_{i,j,k}+G_sT_{i+1,j,k}+G_{s-1}T_{i-1,j,k}+$$
$$+2ET_{i,j\pm1,k}+C', \tag{5-75}$$

where

$$A'=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_z^{(j)}c_j\gamma_j\Big(1-2M_z^{(j)}-2M_x^{(j)}-2N_x^{(j)}M_x^{(j)}+$$

$$+\frac{1}{2}\frac{q_v^{(j)}b_v^{(j)}}{c_j\gamma_j}l_{\overline{n}}\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\overline{n})}\right]-\frac{1}{2}H_jl_{\overline{n}}\Big);$$

$$C'=\frac{1}{D}\sum_{j=s-1}^{s}\frac{1}{2}h_z^{(j)}c_j\gamma_j\Big(2N_x^{(j)}M_x^{(j)}\varphi_{j(z)}+$$

$$+\frac{q_v^{(j)}d_v^{(j)}}{c_j\gamma_j}l_{\overline{n}}\exp\left[-m_v^{(j)}\tau_{k+1/2}^{(\overline{n})}\right]+H_j\chi_jl_{\overline{n}}\Big);$$

$$N_x^{(j)}=\frac{\alpha_{j(z)}h_x}{\lambda_j}.$$

Algorithms (5-74) and (5-75) introduce nothing new to the stability conditions introduced earlier; therefore, we will not write them out.

5. Corner points. The temperature at corner points (for example at the intersection of the boundaries of sections 1 and 2, at the points of inside corners, projections, etc.) can be assumed equal to the arithmetic mean value of temperatures extrapolated to this point using each of the sides of the angle.

The formulas presented above for calculation of temperature on the surface of the body with boundary conditions of the third kind can also be produced from the following considerations.

Let us study the boundary of a body parallel to the OZ axis (Figure 5-15).

Obviously

$$\frac{T_{0,j+1,k}-T_{0,j,k}}{h_x}=\frac{\partial T(0,z,\tau)}{\partial x}+\frac{h_x}{2}\frac{\partial^2 T(0,z,\tau)}{\partial x^2}+O(h_x^2).$$

From the heat conductivity equation (5-51) it follows that:

$$\frac{\partial^2 T(0,z,\tau)}{\partial x^2}=\frac{1}{a}\frac{\partial T(0,z,\tau)}{\partial \tau}-\frac{\partial^2 T(0,z,\tau)}{\partial z^2}+$$
$$+\frac{1}{a}H(T-\chi)-\frac{1}{ac\gamma}q_v(d_v+b_vT)\exp[-m_vt_s].$$

Then

$$\frac{T_{0,j+1,k}-T_{0,j,k}}{h_x}-\frac{h_x}{2}\Big(\frac{1}{a}\frac{\partial T(0,z,\tau)}{\partial \tau}-\frac{\partial^2 T(0,z,\tau)}{\partial z^2}+$$
$$+\frac{1}{a}H(T-\chi)-\frac{1}{ac\gamma}q_v(d_v+b_vT)\exp[-m_vt_s]=$$
$$=\frac{\partial T(0,z,\tau)}{\partial x}+O(h_x^2).$$

But $\quad\dfrac{\partial T(0,z,\tau)}{\partial \tau}\approx\dfrac{T_{0,j,k+1}-T_{0,j,k}}{l};$

$$\frac{\partial^2 T(0,z,\tau)}{\partial z^2}\approx\frac{T_{0,j+1,k}-2T_{0,j,k}+T_{0,j-1,k}}{h_z^2}+O(h_z^2);$$

$$T\approx\frac{T_{1,j,k+1}+T_{1,j,k}}{2};$$

$$\frac{\partial T(0,z,\tau)}{\partial x}=-\frac{\alpha_z}{\lambda}(\varphi_z(\tau_{k+1/2})-T_{0,j,k})$$ (boundary condition).

This also leads to an algorithm which coincides with (5-70).

It is not difficult to see that the order of approximation by this algorithm is $O(h_x^2 + h_z^2)$.

In addition to the formulas presented in the previous and present sections, approximating the boundary conditions and interface conditions, we can also recommend formulas which follow from representation of a derivative with respect to a coordinate as

$$\frac{\partial T_{\Gamma, k}}{\partial z} = \pm \frac{3T_{\Gamma, k} - 4T_{\Gamma \pm 1, k} + T_{\Gamma \pm 2, k}}{2h} + O(h^2).$$

In this case, the values of temperature at the boundary points are determined by the algorithm

$$T_{\Gamma, k} = g_0 + g_1 T_{\Gamma \mp 1, k} + g_2 T_{\Gamma \mp 2, k}, \qquad (5\text{-}76)$$

where, depending on the type of boundary conditions:

	First Kind	Second kind	Third Kind
	$g_0 = \varphi(\tau)$	$\frac{2}{3}\frac{h}{\lambda}\eta(\tau)$	$\frac{2\psi(\tau)}{2h + \frac{\lambda}{\alpha}}$
	$g_1 = 0$	$\frac{4}{3}$	$\frac{4\frac{\lambda}{\alpha}}{2h + \frac{\lambda}{\alpha}}$
	$g_2 = 0$	$-\frac{1}{3}$	$-\frac{\frac{\lambda}{\alpha}}{2h + \frac{\lambda}{\alpha}}$

Here $\phi(\tau)$ is the surface temperature (boundary condition of the first kind); $\eta(\tau)$ is the specific heat flux to the surface (boundary condition of the second kind); $\psi(\tau)$ is the ambient temperature (boundary condition of the third kind); h is the step along the coordinate; λ is the heat conductivity factor.

The interface conditions are written as

$$T^{(s)}_{\Gamma, k} = T^{(s-1)}_{\Gamma, k} = \Pi_{s, s-1}(4T^{(s)}_{\Gamma+1, k} - T^{(s)}_{\Gamma+2, k}) + \Pi_{s-1, s}(4T^{(s-1)}_{\Gamma-1, k} - T^{(s-1)}_{\Gamma-2, k}), \qquad (5\text{-}77)$$

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CALCULATION OF TEMPERATURE FIELDS IN CONCRETE HYDRAULIC STRUCTU--ETC(U)

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where

$$\Pi_{s,s-1} = \frac{\lambda_s}{3\left(\lambda_s + \lambda_{s-1}\frac{h_s}{h_{s-1}}\right)}; \quad \Pi_{s-1,s} = \frac{\lambda_{s-1}}{3\left(\lambda_{s-1} + \lambda_s\frac{h_{s-1}}{h_s}\right)}.$$

For boundary conditions of the third kind, approximated in accordance with (5-76), P. Price and M. Slack [164] produced the stability condition

$$\frac{\alpha h}{\lambda} \geqslant 1. \qquad (5\text{-}78)$$

Based on a comparison of the stability conditions (5-63) presented earlier with (5-78), we can see that the former places less limitations on the time step ℓ with low values of α and h, whereas the latter requires larger values of α and h. Consequently, joint utilization of various algorithms for boundary conditions and interface conditions allows economical performance of calculations of temperature fields.

Thus, in calculating two-dimensional temperature fields for masses of $\bar{n}$ blocks using an explicit six-point finite difference plan, time step $\ell_{\bar{n}}$ should satisfy the conditions

$$l_{\bar{n}} \leqslant \frac{1}{a_s\left[\frac{2}{(h_z^{(s)})^2} + \frac{1}{(h_x^{(s)})^2}\left(2 + N_x^{(s)}\right)\right]} \quad (s = 0, 1, 2, \ldots, \bar{n});$$

$$l_{\bar{n}} \leqslant \frac{1}{a_{\bar{n}}\left[\frac{2}{(h_x^{(\bar{n})})^2} + \frac{1}{(h_z^{(\bar{n})})^2}\left(2 + N_z^{(\bar{n})}\right)\right]},$$

if the temperature at the boundary is calculated by the algorithms (5-70), (5-72), (5-74), (5-75), and

$$l_{\bar{n}} \leqslant \frac{1}{2a_s\left[\frac{1}{(h_z^{(s)})^2} + \frac{1}{(h_x^{(s)})^2}\right]} \quad (s = 0, 1, 2, \ldots, \bar{n});$$

$$N_x^{(s)} = \frac{\alpha_s(x)\, h_x^{(s)}}{\lambda_s} \geqslant 1 \quad (s = 0, 1, 2, \ldots, \bar{n});$$

$$N_z^{(\bar{n})} = \frac{\alpha_{\bar{n}}(x)\, h_z^{(\bar{n})}}{\lambda_{\bar{n}}} \geqslant 1,$$

if the temperature at the boundary is calculated by the algorithms (5-76), (5-77).

In order to increase the accuracy of calculation in the first time steps, the initial temperature at the interface of the blocks, by analogy with the thermal state of the contact boundary between two semiinfinite rods [69], should be taken as

$$T_{\Gamma,0} = T^{(s)} + (T^{(s-1)} + T^{(s)}) \left[1 + \sqrt{\frac{\lambda_s c_s \gamma_s}{\lambda_{s-1} c_{s-1} \gamma_{s-1}}} \right]^{-1},$$

where $T^{(p)}$ (p = s, s-1) is the initial temperature of the block.

Based on the relationships presented, approximating the differential equations, the boundary conditions and conditions of conjugation at the interface of the bodies, the All-Union Scientific Research Institute for Hydraulogy imeni B. Ye. Vedeneyev has written computer programs for one-dimensional and two-dimensional temperature fields in concrete masses constructed in individual blocks.

As was indicated earlier in the problem of a discretely growing mass, the time step, defined from the corresponding stability conditions, may be variable. Therefore, in placing each subsequent block we define $\ell_{\overline{n}}$, which is compared with the previous value and subsequent calculation is performed using the lesser value of ℓ.

In addition to the stability condition, the time step ℓ in each stage follows the conditions of multiplicity of time sectors of the generalized heat liberation intensity function, as well as the requirement that in the period when heat liberation occurs within the concrete, $\ell \leq 6$ hr.

The algorithm is so constructed that the base is looked upon as one of the blocks of the mass and is assigned the subscript 0. The temperature mode of the base can be calculated prior to placement of the first block. From the moment $\tau_0 = 0$ on the horizontal surface the boundary conditions of one of the three types outlined above are assigned and calculation is performed up to τ_1 -- the time of placement of the first block.

The parameter of the generalized heat liberation intensity function $q_\nu^{(s)}$, $d_\nu^{(s)}$, $b_\nu^{(s)}$, $m_\nu^{(s)}$, as well as the duration of the time sectors $(\tau_{\nu-1}, \tau_\nu)$ are assigned with the initial information. In addition to this, the heat liberation, which depends only on time, can be represented in tabular form. In this case, the parameters of the heat liberation intensity function $q_\nu^{(s)}$ and $m_\nu^{(s)}$ are determined using a special subroutine by the method outlined in § 2-2.

In order to prepare the initial information concerning the mass being constructed, we must have the concrete pouring schedule and data on each block (height R_s, temperature conductivity factor a_s, heat conductivity factor λ_s, initial temperature of the concrete when it is poured $f_s(z)$, type of boundary conditions at the upper surface and temperature of this surface $\phi_{\bar{n}}(\tau)$ or of the environment $\psi_{\bar{n}}(\tau)$, parameter H_s and ambient temperature $\chi_{\bar{n}}(\tau)$, etc.). However, the program is so constructed that data on the next block to be laid are introduced to the process of calculation when a fixed moment in time is reached.

Thus, it is possible to model by computer the process of construction of the mass in parallel with the actual construction of the structure.

The program of calculation of two-dimensional temperature fields in its standard form was developed for masses of rectangular shape with constant width of a column being constructed. However, analysis showed that this limitation is easy to eliminate. With some additional programming, any geometric shape of concrete masses can be considered which can be inscribed into a rectangle. The program is written so that a rectangular grid is formed using the number of divisions of width and height selected and the temperature at the junctions of this grid is calculated for each time step.

The essence of the additional programming consists in the formation of the required complex contour in the rectangular grid, in which the corresponding boundary conditions are assigned. When necessary, the base is eliminated from calculation.

Supplement. *The extension of the results produced to the case of description* of the functions of intensity of heat liberation in the concrete by the formula of I. D. Zaporozhets

$$q(\tau, T) = q_0 2^{\frac{T-20}{\varepsilon}} \left[1 + A_{20} \int_0^{\tau} 2^{\frac{T-20}{\varepsilon}} d\tau \right]^{-\frac{m}{m-1}}$$

represents no difficulty. To do this, in all calculation formulas for the coefficients A, C and D we must place $q_\nu = 0$, and in the formulas for coefficients C, furthermore, add the term

$$\frac{q_0 l}{c\gamma} 2^{\frac{T_{i,k}-20}{\varepsilon}} \left[1 + A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{i,p}-20}{\varepsilon}} \right]^{-\frac{m}{m-1}}.$$

Coefficient C in this case is not only dependent on the space-time coordinates of the points in the calculation area in which exothermy is observed, but also on the thermal prehistory of each point in space.

Temperature Fields in Concrete Masses in the Process of Construction of Certain Dams

1. Temperature mode of one column of the Krasnoyarsk Power Plant Spillway Dam. The initial data were provided by the Laboratory of Field Investigations of Hydraulic Engineering Structures of the All-Union Scientific Research Institute for Hydrology [6].

Pouring of the column continued for over a year, the block overlap intervals varying from 5 days to several months. Depending on the time of year, the initial temperature of the concrete mixture varied from 7.6 to 19.0 C. The ambient temperature on the horizontal surface was assumed equal to the air temperature in the construction region. In this case, when a tent was placed over the horizontal surface of the block, the ambient temperature was assumed constant. The ambient temperature on the side surfaces was variable over the height of the dam. Due to the blockage of the Yenisey River, the concrete column was partially submerged, the water level and its temperature varying with time. Boundary conditions of the first kind were assumed at surfaces of contact with the water. The thermal protective properties of the deck were considered by introduction of an equivalent heat transfer coefficient, its numerical value, depending on the decking, varying from 0.65 to 3.0 kcal/($m^2 \cdot hr \cdot C$). The nonmetallic deck, upon contact with the water, changed its heat-physical characteristics, which was kept in mind in calculating the equivalent heat transfer coefficient for the submerged portions of the column.

The varying composition of the concrete in individual blocks (grade 500 Portland cement, grade 400 slag-Portland cement, cement content 240 to 315 kg/m^2) naturally, led to a variation in heat liberation over the height of the column. The initial values used were the experimental data on adiabatic heat liberation of concretes of the corresponding compositions. These data were processed by the method outlined in § 2-2.

The dimensions of the column in plan are (9-15) x 11.5 m.

A study was made of the planar temperature field in the middle vertical longitudinal cross section of the column. In order to consider heat exchange from the third (length of column 11.5 m) dimension, terms with parameter H were introduced to the differential equations, the numerical values of which were established on the basis of the model presented in § 5-3.

Figure 5-16 and 5-17 present the calculation data and data from field studies performed under the leadership of E. K. Aleksandrovskaya. As we can see from the figures, the agreement of the results of calculation with field data is satisfactory.

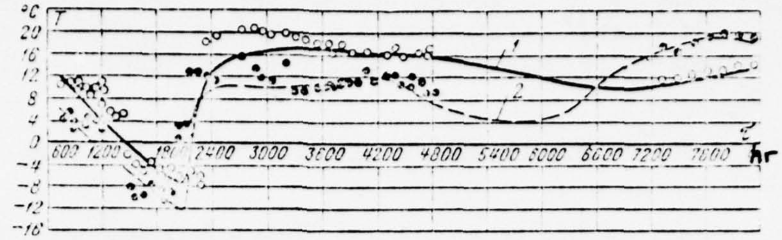


Figure 5-16. Temperature at Points of Installation of Sensors According to Field Data and Results of Calculation (Block 3). 1, Calculation Curve at Point on Middle of Block Axis; 2, Calculation Curve at Point Located 50 cm from Side Surface; ●; O, Field Data; ———, Calculation Curves

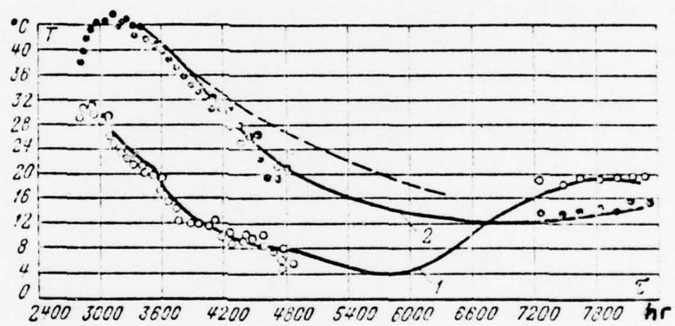


Figure 5-17. Temperature at Points of Installation of Sensors According to Field Data and Calculation Results (Block 6). 1, Calculated Curve at Point on Center of Block Axis; 2, Calculation Curve at Point Located 50 cm from Side Surface; ●; O, Field Data; ———, Calculation Curves

The slight divergence which does occur in the 1000-1500 and 2400-3200 hr ranges (Figure 5-16) can be explained by the following factors. The initial information contained data on the air temperature in a drain cavity from mid-December (2000 hr after the beginning of construction of the mass). The air temperature used in calculations was produced by extrapolation of the curve describing the actual temperature during later periods of time. Furthermore, in order to provide cable leads, near the temperature measuring devices in the third block (by height) a well was emplaced into the depth of the third and fourth blocks. This well was filled with concrete when the fifth block was poured, and its exothermic heating led to an increase in temperature at the points of measurement in the third block.

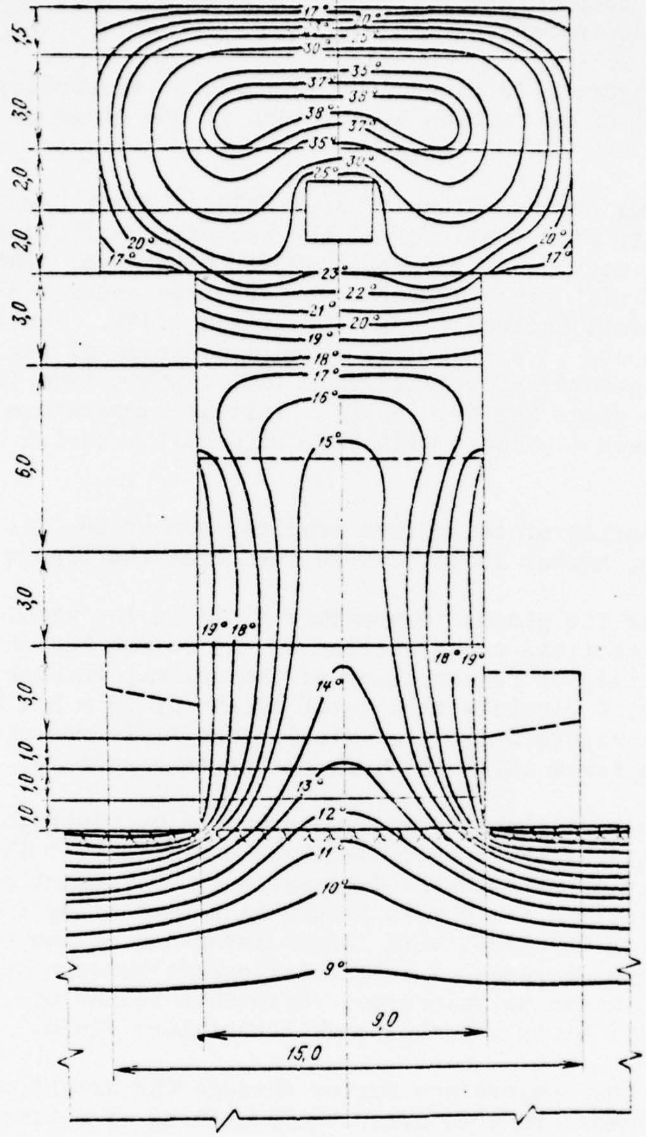


Figure 5-18. Temperature Field of
One Column of the Krasnoyarsk Power
Plant Spillway Dam

Figure 5-17 presents two calculated temperature curves for a point midway up the axis of the 6th block. The dotted curve was produced on the assumption that heat exchange stopped after the third measurement (H = 0). The solid curve considers this heat exchange. Our attention is drawn by the importance of considering heat exchange following the third measurement in

calculating planar temperature fields in concrete columns, the plan dimensions of which are comparable.

The isotherms of Figure 5-18 give us an idea of the temperature field in the middle vertical cross section of a column at one stage of its construction.

2. Temperature mode of one column of Bratsk Power Plant Dam. Figure 5-19 presents the results of calculation of the temperature field of a concrete column in the 30th section of the Bratsk Power Plant Dam. The calculated temperatures agree with field measurement performed under the leadership of Doctor of Technical Sciences S. Ya. Eydel'man [145]. The dimensions of the column in plan are 13.8 x 15.0 m. The composition of the concrete of the blocks is: slag-Portland cement grade 400, content from 170 to 240 kg/m^3 concrete; concrete grade 200 V8, 100 V2. Initial temperature of concrete mixture 5-10 C. Deck -- wooden with effective heat transfer coefficient 1.5 kcal/(m^2·hr·C).

The schedule of pouring of the column studied, number 30-III, and of a neighboring column, number 30-II, can be seen from the figure.

A study was made of the planar temperature field in the middle (through the 15.0 m dimension) vertical cross section of the column beginning in December of 1960 (beginning of construction of the column) through March of 1961 (by this time, 6 blocks with a total height of 18 m had been poured). As we can see from Figure 5-19, the calculations performed yield a picture of the temperature field which is close to the true picture.

3. Temperature mode of concrete mass constructed by "Toktogul'skaya" method (dam of Toktogul'skaya Hydroelectric Power Plant). The Toktogul'skaya method of construction of concrete dams calls for placement of the concrete in layers -- in blocks 0.5-1.0 m in height with compacting immediately through their entire thickness with large dimensions of the block in plan (up to 32 x 75 m and larger). A height difference between neighboring sectors of 0.5-1.0 m can be tolerated. With this technology, most of the concrete poured will be in a homogeneous temperature field.

Figure 5-20 shows the temperature curves through the height of the concrete mass at various moments in time after the beginning of construction.

The concrete mass was poured in blocks 0.75 and 1.0 m high. Between 24 April and 7 November, blocks 0.75 m high were poured with an overlap interval of 6 days, during the remaining portion of the year -- 1.0 m high with an overlap interval of 8 days. Pouring of the concrete mass began on 15 April. The temperature of the concrete mixture was constant throughout the year, at 12 C.

The basic measure used to regulate the temperature of the concrete mass was pouring of water over the surface. The water temperature corresponded to the figures provided by the Central Construction Laboratory of the

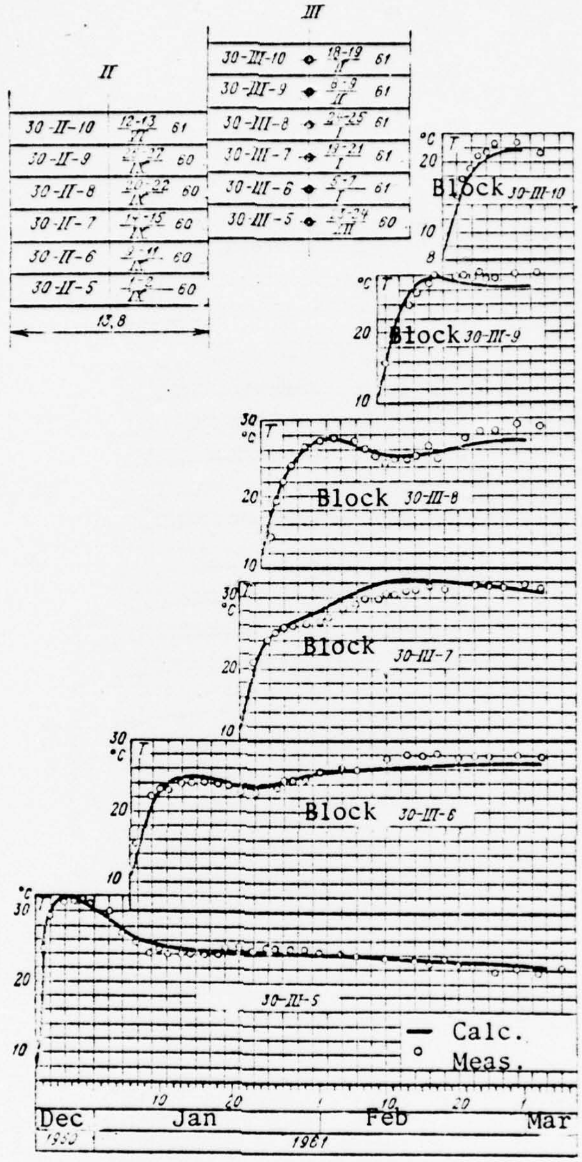


Figure 5-19. Temperature at Points of Installation of Sensors According to Field Data and Results of Calculation (Concrete Column in 30th Section of Bratsk Power Plant Dam). O, Field Data; ———, Calculation Data

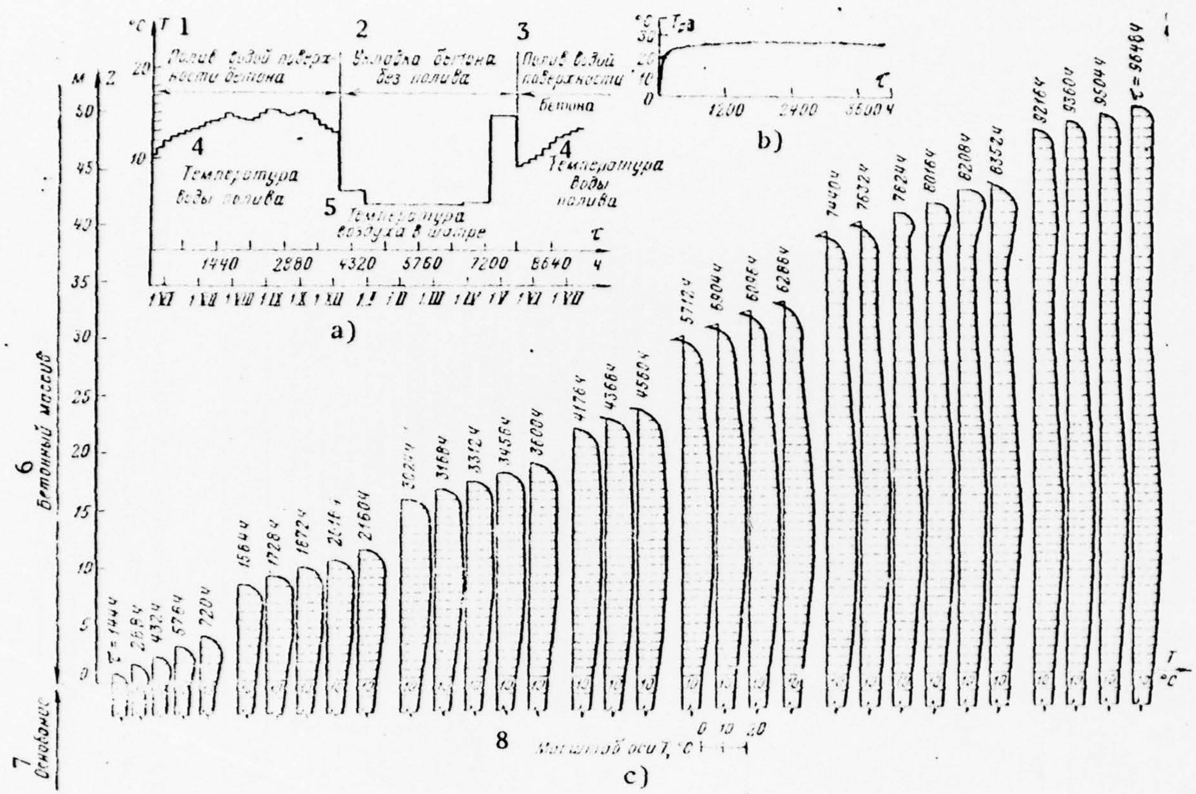


Figure 5-20. Temperature Field of a Concrete Mass Constructed by the Toktogul'skaya Method. a, Graph of Temperature of Water Covering Concrete Surface and Air in Tent; b, Graph of Adiabatic Temperature Rise; c, Temperature of Concrete Mass; 1, Water Flooding of Concrete Surface; 2, Concrete Poured without Flooding; 3, Water Flooding of Surface; 4, Water Temperature; 5, Air Temperature in Tent; 6, Concrete Mass; 7, Base; 8, Scale of T Axis, C

Toktogul'skaya Hydroelectric Power Plant. Surface flooding was used from 24 April through 7 November. The concrete was poured beneath a tent. The air temperature in the tent was 5 C in December-February, equal to the air temperature in the construction region during the remainder of the year (see Table 2-12).

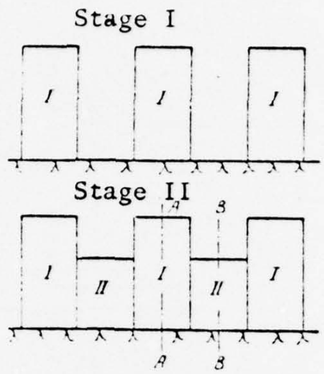


Figure 5-21. Diagram of Construction Version with Concreting of Blocks "in Gaps"

4. Temperature field of a concrete mass with separate construction of alternating columns and subsequent concreting of the spaces by pouring blocks "in the gaps." This method of construction is characteristic for column construction of structures with pouring of blocks.

The construction version in question is performed in two stages, schematically diagrammed in Figure 5-21.

It is not difficult to see that the lines AA and BB are lines of symmetry. This fact was kept in mind in formulation of the boundary conditions of the problem.

We studied the two-dimensional (planar) temperature field, introducing terms considering heat exchange in the third dimension to the heat conductivity equation.

It was assumed that the concrete blocks differed in height, and also differed in the initial temperature of placement of the blocks, growth rate of the primary columns and intermediate columns. The ambient temperature was constant over the horizontal surfaces of the blocks but varied harmonically on the side surfaces (in the third dimension).

The results of calculation of the temperature field during various stages of the process of construction are presented in Figure 5-22.

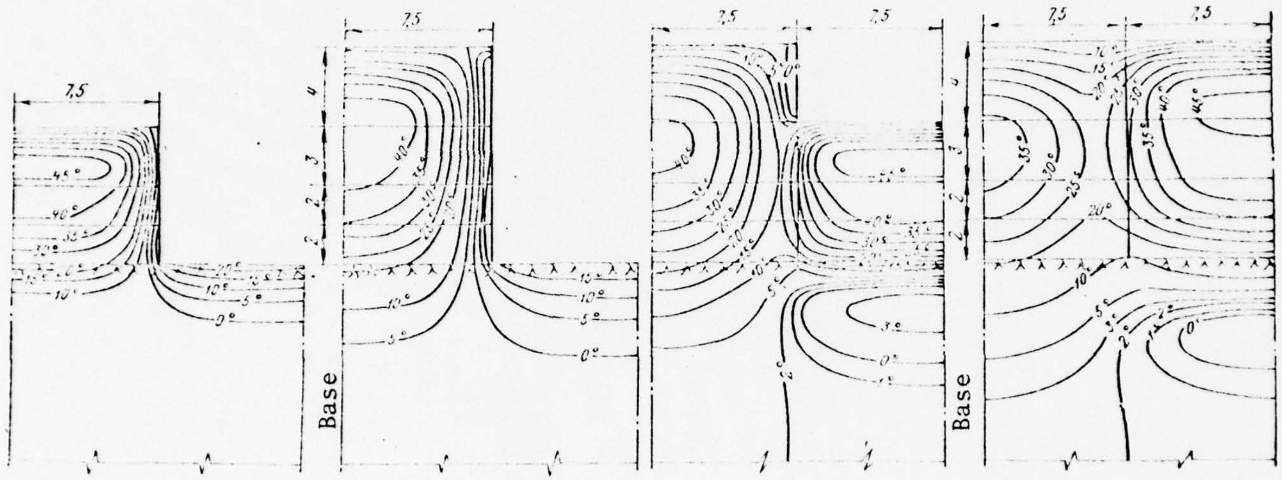


Figure 5-22. Temperature Field of a Mass with Separate Construction of Alternating Columns and Subsequent Concreting of the Spaces "in the Gaps"

5-5. Methods of Calculation of Temperature Fields of "High" Concrete Blocks

Recently, researchers, planners and constructors have shown increased interest in the construction of masses using "high" blocks. This method of working involves continuous pouring of blocks to heights of 20 m and more.

The primary advantage of continuous concreting in "high" blocks is a significant decrease in consumption of labor for treatment of horizontal construction seams and the possibility of significantly increasing the rate of placement of concrete, which is particularly important in the construction of concrete dams in areas with severe climatic conditions.

The construction of water engineering structures in "high" blocks has been most widely used in Canada [158]. Between 1931 and 1958, this method was used there for the construction of 15 massive concrete dams up to 72 m in height. The concrete used is of ordinary composition, thermal control measures consisted of heat insulation of the surfaces, regular construction of neighboring blocks, some decrease in heat liberation of the concrete by selection of proper cements and preliminary cooling of the concrete mixture during the summer by the use of cold water.

In the Soviet Union, for example in the construction of the Bukhtarminskaya, Bratsk and Krasnoyarsk Power Plants, individual structural elements were poured in blocks 6-9 m high.

There are certain difficulties involved in simply adopting the Canadian experience to domestic water engineering construction practice. The climatic conditions of Siberia and the Far East, the primary regions of water power engineering of the USSR, are incomparably more severe than in Canada, and the concrete dams currently under construction and in planning are significantly higher than the Canadian dams.

We must therefore perform special investigations, one of the main stages of which is the selection and development of a basis for methods of regulation of the temperature mode of the concrete masses to provide for a monolithic structure.

As was shown in § 5-1, in hydraulic engineering construction practice, the blocks are poured in layers 30-50 cm high, the layers being poured at intervals of 1.5-2 hr. Consequently, pouring of a high block, for example 20 m high, requires two to three days. Under these conditions, we cannot ignore the prehistory of formation of the thermal state, assuming that the block was put in place instantaneously.

In principle, the method outlined above for calculation of the temperature field of masses poured in discrete blocks is suitable for determination of the thermal state of high blocks; we must only consider the layers, rather than the blocks, as the discrete units. However, in this case the number

of discrete units is significantly increased and the calculations become quite cumbersome.

As studies have shown, at the concreting rates used in the USSR, the temperature fields of high blocks can be calculated using a model of even and continuous growth of the height of the masses. This model was used as the basis for analytic studies in this section, as a result of which a method is suggested for calculation of the temperature fields of high concrete blocks both in the stage of construction and in the period of conservation, i.e., immediately after the process of construction is stopped.

Calculation of the Temperature Field in a Continuously Growing Concrete Mass

Three-dimensional temperature field.

Construction period. A study is made of a semilimited column of cross section 2L x 2D with initial temperature T_0. At $\tau = 0$, continuous growth is begun at constant rate b. The initial temperature of the material placed is T_b. Due to hydration of the cement, heat liberation occurs in the growing concrete portion of the column, the intensity of which is determined by the expression $q_0 \, e^{-m\tau}$. Heat exchange of the column with the environment is by convection (boundary condition of the third kind). The heat-physical characteristics of the base and concrete are identical.

Let us place the coordinate origin at the division boundary of the growing concrete portion of the column and the base.

The problem is then formulated as follows.

The differential equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{q_0}{c\gamma} e^{-m\left(\tau - \frac{z}{b}\right)} \varepsilon\left(\tau - \frac{z}{b}\right)$$
$$(-\infty < z < b\tau, \ -L < x < L, \ -D < y < D, \ \tau > 0). \qquad (5\text{-}79)$$

The initial condition

$$T(z, x, y, 0) = T_0 \ (-\infty < z \leqslant 0, \ -L \leqslant x \leqslant L, \ -D \leqslant y \leqslant D). \qquad (5\text{-}80)$$

The boundary conditions

$$\frac{\partial T(b\tau, x, y, \tau)}{\partial z} = H_1[\Phi_1 - T(b\tau, x, y, \tau)];$$

$$\frac{\partial T(-\infty, x, y, \tau)}{\partial z} = 0;$$

$$\frac{\partial T(z, L, y, \tau)}{\partial x} = h_2[T_2 - T(z, L, y, \tau)];$$

$$\frac{\partial T(z, 0, y, \tau)}{\partial x} = 0 \text{ (condition of symmetry);} \qquad (5\text{-}81)$$

$$\frac{\partial T(z, x, D, \tau)}{\partial y} = h_2[T_2 - T(z, x, D, \tau)];$$

$$\frac{\partial T(z, x, 0, \tau)}{\partial y} = 0 \text{ (condition of symmetry).}$$

Here $\varepsilon(\tau - (z/b))$ is a unit function, equal to

$$\varepsilon\left(\tau - \frac{z}{b}\right) = \begin{cases} 1 \text{ where } \tau - \frac{z}{b} < \tau, \\ 0 \text{ where } \tau - \frac{z}{b} > \tau. \end{cases}$$

Introduction of the unit function ε allows us to describe the heat liberation in the growing portion of the column and the absence of heat liberation in the base quite simply (ε is equal to unity in the area $x > 0$ and to 0 in the area $x < 0$).

The first boundary condition (5-81) is produced from the equation for thermal balance at the corresponding boundary of the column. The initial temperature of the material poured T_b is looked upon as a factor leading to additional heat flux to the horizontal surface of the mass, equal to $bc\gamma(T_b - T)$.

The thermal balance equation in this case becomes:

$$\frac{\partial T(b\tau, x, y, \tau)}{\partial z} = h_1[T_1 - T(b\tau, x, y, \tau)] + \frac{bc\gamma}{\lambda}[T_6 - T(b\tau, x, y, \tau)],$$

which leads to the first condition of (5-81), where

$$H_1 = h_1 + \frac{b}{a};$$

$$\Phi_1 = \frac{h_1 T_1 + \frac{b}{a} T_b}{h_1 + \frac{b}{a}}.$$

Let us assume

$$\vartheta = T_2 - T,$$

and apply to (5-79)-(5-81) a double cosine transform

$$\bar{\bar{\vartheta}} = \int_0^L \int_0^D \vartheta \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \, dx \, dy,$$

where μ_p and κ_r are the roots of the transcendental equations

$$\cot \mu_p = \frac{\mu_p}{\mathrm{Bi}_1}; \ \cot \varkappa_r = \frac{\varkappa_r}{\mathrm{Bi}_2}; \ \mathrm{Bi}_1 = h_2 L; \ \mathrm{Bi}_2 = h_2 D.$$

We produce

$$\frac{\partial \bar{\bar{\vartheta}}}{\partial \tau} = a \frac{\partial^2 \bar{\bar{\vartheta}}}{\partial z^2} - a\left(\frac{\mu_p^2}{L^2} + \frac{\varkappa_r^2}{D^2}\right)\bar{\bar{\vartheta}} - \frac{q_0}{c\gamma} N_p M_r e^{-m\left(\tau - \frac{z}{b}\right)} \varepsilon\left(\tau - \frac{z}{b}\right);$$

$$\bar{\bar{\vartheta}}(z, 0) = (T_2 - T_0) N_p M_r;$$

$$\frac{\partial \bar{\bar{\vartheta}}(b\tau, \tau)}{\partial z} = H_1 [(T_2 - \Phi_1) N_p M_r - \bar{\bar{\vartheta}}(b\tau, \tau)];$$

$$\frac{\partial \bar{\bar{\vartheta}}(-\infty, \tau)}{\partial z} = 0.$$

Here

$$N_p = \int_0^L \cos \mu_p \frac{x}{L} \, dx;$$

$$M_r = \int_0^D \cos \varkappa_r \frac{y}{D} \, dy.$$

The substitution

$$\bar{\bar{\vartheta}} = (T_2 - \Phi_1) N_p M_r - \bar{\bar{\vartheta}}$$

gives:

$$\frac{\partial\bar{\bar{\vartheta}}}{\partial\tau}=a\frac{\partial^2\bar{\bar{\vartheta}}}{\partial z^2}-a\left(\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}\right)\bar{\bar{\vartheta}}+\frac{q_0}{c\gamma}N_pM_r\times$$
$$\times e^{-m\left(\tau-\frac{z}{b}\right)}\varepsilon\left(\tau-\frac{z}{b}\right)+a\left(\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}\right)(T_2-\Phi_1)N_pM_r;$$
$$\bar{\bar{\vartheta}}(z,0)=(T_0-\Phi_1)N_pM_r;$$
$$\frac{\partial\bar{\bar{\vartheta}}(b\tau,\tau)}{\partial z}=-H_1\bar{\bar{\vartheta}}(b\tau,\tau);$$
$$\frac{\partial\bar{\bar{\vartheta}}(-\infty,\tau)}{\partial z}=0.$$

Following [35, 170], we perform substitution of variables

$$\xi=\tau-\frac{z}{b};\ \tau=t,$$

equivalent to introduction of a moving system of coordinates, the origin of which is located on the upper horizontal surface of the column, while the 0ξ axis is directed into the depth of the mass. We note that the new coordinate ξ is always positive ($\xi > 0$).

We then have:

$$\frac{\partial\bar{\bar{\vartheta}}}{\partial t}=\frac{a}{b^2}\frac{\partial^2\bar{\bar{\vartheta}}}{\partial\xi^2}-\frac{\partial\bar{\bar{\vartheta}}}{\partial\xi}-a\left(\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}\right)\bar{\bar{\vartheta}}+$$
$$+\frac{q_0}{c\gamma}N_pM_r\varepsilon(\xi)+a\left(\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}\right)(T_2-\Phi_1)N_pM_r;$$
$$\bar{\bar{\vartheta}}(\xi,0)=(T_0-\Phi_1)N_pM_r;$$
$$\frac{\partial\bar{\bar{\vartheta}}(0,t)}{\partial\xi}=bH_1\bar{\bar{\vartheta}}(0,t);\ \frac{\partial\bar{\bar{\vartheta}}(\infty,t)}{\partial\xi}=0.$$

The transform

$$\bar{\bar{\vartheta}}=u\exp\left[\frac{b^2}{2a}\xi-S_{pr}at\right],$$

where

$$S_{pr}=\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}+\frac{b^2}{4a^2},$$

brings the problem to the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{a}{b^2}\frac{\partial^2 u}{\partial \xi^2} + F(\xi, t);\\
&u(\xi, 0) = \Psi(\xi);\\
&\frac{\partial u(0, t)}{\partial \xi} = hu(0, t);\\
&\frac{\partial u(\infty, t)}{\partial \xi} = 0,
\end{aligned}
\tag{5-82}
$$

where

$$
\begin{aligned}
F(\xi, t) = {} & a\left(S_{pr} - \frac{b^2}{4a^2}\right)(T_2 - \Phi_1) N_p M_r \exp\left[-\frac{b^2}{2a}\xi + S_{pr}at\right] + \\
& + \frac{q_0}{c\gamma} N_p M_r \varepsilon(\xi) \exp[-G\xi + S_{pr}at];\\
\Psi(\xi) = {} & (T_0 - \Phi_1) N_p M_r \exp\left[-\frac{b^2}{2a}\xi\right];\\
h = {} & bH_1 - \frac{b^2}{2a} = bh_1 + \frac{b^2}{2a};\\
G = {} & m + \frac{b^2}{2a};
\end{aligned}
$$

$$
\varepsilon(\xi) = \begin{cases} 1 & \text{where } \xi < t, \\ 0 & \text{where } \xi > t. \end{cases}
$$

The solution of the edge problem (5-82) can be produced, for example, by means of a Green function.

The final solution of the problem of the spatial temperature field of a continually growing column is:

$$
T = T_2 + \sum_{p=1}^{\infty}\sum_{r=1}^{\infty} A_p B_r \cos\mu_p \frac{x}{L} \cos\varkappa_r \frac{y}{D} V_{pr}(\xi, t). \tag{5-83}
$$

Here

$$
\begin{aligned}
V_{pr}(\xi, t) = {} & \frac{1}{2}(T_0 - T_2)\left(\exp\left[-\left(S_{pr} - \frac{b^2}{4a^2}\right)at\right] \times \right.\\
& \times \operatorname{erfc}\left[\frac{b(t-\xi)}{2\sqrt{at}}\right] - \frac{h + \frac{b^2}{2a}}{h - \frac{b^2}{2a}} \exp\left[\left(S_{pr} - \frac{b^2}{4a^2}\right)at + \right.
\end{aligned}
$$

$$+\frac{b^2}{a}\xi\Big]\operatorname{erfc}\Big[\frac{b(t+\xi)}{2\sqrt{at}}\Big]\Big)-\frac{q_0}{2ac\gamma\Big(S_{pr}-\frac{G^2}{b^2}\Big)}\Big\{\exp\Big[-\Big(S_{pr}-\frac{G^2}{b^2}\Big)\times$$

$$\times at-\Big(G-\frac{b^2}{2a}\Big)\xi\Big]\operatorname{erfc}\Big[\frac{G}{b}\sqrt{at}-\frac{b\xi}{2\sqrt{at}}\Big]-$$

$$-\exp\Big[-\Big(S_{pr}-\frac{G^2}{b^2}\Big)at-\Big(G-\frac{b^2}{2a}\Big)\xi\Big]\operatorname{erfc}\Big[\frac{G}{b}\sqrt{at}+\frac{b(t-\xi)}{2\sqrt{at}}\Big]-$$

$$-\frac{h+G}{h-G}\exp\Big[-\Big(S_{pr}-\frac{G^2}{b^2}\Big)at+\Big(G+\frac{b^2}{2a}\Big)\xi\Big]\operatorname{erfc}\Big[\frac{G}{h}\sqrt{at}+\frac{b\xi}{2\sqrt{\beta t}}\Big]+$$

$$+\frac{h+G}{h-G}\exp\Big[-\Big(S_{pr}-\frac{G^2}{b^2}\Big)at+\Big(G+\frac{b^2}{2a}\Big)\xi\Big]\operatorname{erfc}\Big[\frac{G}{b}\sqrt{at}+$$

$$+\frac{b(t+\xi)}{2\sqrt{at}}\Big]-\frac{2h}{h-G}\exp\Big[-\Big(S_{pr}-\frac{h^2}{b^2}\Big)at+\Big(h+\frac{b^2}{2a}\Big)\xi+$$

$$+(h-G)t\Big]\operatorname{erfc}\Big[\frac{h}{b}\sqrt{at}+\frac{b(t+\xi)}{2\sqrt{at}}\Big]+\frac{1}{2\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}-\frac{h}{b}\Big)^{-1}\times$$

$$\times\Big(\sqrt{S_{pr}}+\frac{h}{b}\Big)\Big(\sqrt{S_{pr}}+\frac{G}{b}\Big)\exp\Big[\Big(b\sqrt{S_{pr}}+\frac{b^2}{2a}\Big)\xi+$$

$$+(b\sqrt{S_{pr}}-G)t\Big]\operatorname{erfc}\Big[\sqrt{S_{pr}at}+\frac{b(t+\xi)}{2\sqrt{at}}\Big]+$$

$$+\frac{1}{2\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}+\frac{h}{b}\Big)^{-1}\Big(\sqrt{S_{pr}}-\frac{h}{b}\Big)\Big(\sqrt{S_{pr}}-\frac{G}{b}\Big)\times$$

$$\times\exp\Big[-\Big(b\sqrt{S_{pr}}-\frac{b^2}{2a}\Big)\xi-(b\sqrt{S_{pr}}+G)t\Big]\operatorname{erfc}\Big[-\sqrt{S_{pr}at}+$$

$$+\frac{b(t+\xi)}{2\sqrt{at}}\Big]-\frac{1}{2\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}+\frac{G}{b}\Big)\exp\Big[-\Big(b\sqrt{S_{pr}}-$$

$$-\frac{b^2}{2a}\Big)\xi+(b\sqrt{S_{pr}}-G)t\Big]\operatorname{erfc}\Big[-\sqrt{S_{pr}at}-\frac{b(t-\xi)}{2\sqrt{at}}\Big]-$$

$$-\frac{1}{2\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}-\frac{G}{b}\Big)\exp\Big[\Big(b\sqrt{S_{pr}}+\frac{b^2}{2a}\Big)\xi-(b\sqrt{S_{pr}}+G)t\Big]\times$$

$$\times\operatorname{erfc}\Big[\sqrt{S_{pr}at}-\frac{b(t-\xi)}{2\sqrt{at}}\Big]-\varepsilon\Big(2\exp\Big[-\Big(G-\frac{b^2}{2a}\Big)\xi\Big]-$$

$$-\frac{1}{\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}-\frac{G}{b}\Big)\exp\Big[\Big(b\sqrt{S_{pr}}+\frac{b^2}{2a}\Big)\xi-(b\sqrt{S_{pr}}+G)t\Big]-$$

$$-\frac{1}{\sqrt{S_{pr}}}\Big(\sqrt{S_{pr}}+\frac{G}{b}\Big)\exp\Big[-\Big(b\sqrt{S_{pr}}-\frac{b^2}{2a}\Big)\xi+$$

$$+\Big(b\sqrt{S_{pr}}-G\Big)t\Big]\Big]\Big\}+\Big(\frac{h}{h-\frac{b^2}{2a}}\Big[T_0-\Phi_1+(T_2-\Phi_1)\times$$

$$\times\Big(S_{pr}-\frac{h^2}{b^2}\Big)^{-1}\Big(S_{pr}-\frac{b^2}{4a^2}\Big)\Big]-\frac{q_0h}{ac\gamma(h-G)}\Big(S_{pr}-\frac{h^2}{b^2}\Big)^{-1}\Big)\times$$

$$\times \exp\left[-\left(S_{pr}-\frac{b^2}{4a^2}\right)at+\left(h+\frac{b^2}{2a}\right)\xi\right]\times$$

$$\times \operatorname{erfc}\left[\frac{h}{b}\sqrt{at}+\frac{b\xi}{2\sqrt{at}}\right]+\left((T_2-\Phi_1)\left[\frac{b}{4a\sqrt{S_{pr}}}-\right.\right.$$

$$-\frac{h}{2\sqrt{S_{pr}}}\left(h-\frac{b^2}{2a}\right)^{-1}\left(\sqrt{S_{pr}}+\frac{b}{2a}\right)-\frac{h}{2\sqrt{S_{pr}}}\left(\sqrt{S_{pr}}-\frac{h}{b}\right)^{-1}\times$$

$$\times\left(h-\frac{b^2}{2a}\right)^{-1}\left(S_{pr}-\frac{b^2}{4a^2}\right)\Big]+\frac{q_0(h+G)}{2ac\gamma}\left(\sqrt{S_{pr}}-\frac{h}{b}\right)^{-1}\times$$

$$\times\left(S_{pr}-\frac{G^2}{b^2}\right)^{-1}\Big)\exp\left[\left(b\sqrt{S_{pr}}+\frac{b^2}{2a}\right)\xi\right]\operatorname{erfc}\left[\sqrt{S_{pr}at}+\frac{b\xi}{2\sqrt{at}}\right]-$$

$$-\left((T_2-\Phi_1)\left[\frac{b}{4a\sqrt{S_{pr}}}+\frac{h}{2\sqrt{S_{pr}}}\left(h-\frac{b^2}{2a}\right)^{-1}\left(\sqrt{S_{pr}}-\frac{b}{2a}\right)+\right.\right.$$

$$+\frac{h}{2\sqrt{S_{pr}}}\left(\sqrt{S_{pr}}+\frac{h}{b}\right)^{-1}\left(h-\frac{b^2}{2a}\right)^{-1}\left(S_{pr}-\frac{b^2}{4a^2}\right)\Big]+$$

$$+\frac{q_0(h+G)}{2ac\gamma}\left(\sqrt{S_{pr}}+\frac{h}{b}\right)^{-1}\left(\sqrt{S_{pr}}-\frac{G^2}{b^2}\right)^{-1}\Big)\times$$

$$\times\exp\left[-\left(b\sqrt{S_{pr}}-\frac{b^2}{2a}\right)\xi\right]\operatorname{erfc}\left[-\sqrt{S_{pr}at}+\frac{b\xi}{2\sqrt{at}}\right]\Big\};$$

$$A_p=\frac{(-1)^{p+1}2\mathrm{Bi}_1\sqrt{\mathrm{Bi}_1^2+\mu_p^2}}{\mu_p(\mathrm{Bi}_1^2+\mathrm{Bi}_1+\mu_p^2)};$$

$$B_r=\frac{(-1)^{r+1}2\mathrm{Bi}_2\sqrt{\mathrm{Bi}_2^2+\varkappa_r^2}}{\varkappa_r(\mathrm{Bi}_2^2+\mathrm{Bi}_2+\varkappa_r^2)}. \qquad (5\text{-}84)$$

The remaining symbols are the same as before.

Period of conservation. The period of conservation refers to that time τ' after completion of the process of erection. Therefore, we are speaking of the solution of the following problem.

The differential equation

$$\frac{\partial T'}{\partial \tau'}=a\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{q_0}{c\gamma}\exp\left[-m\left(\tau_B-\frac{z}{b}\right)-\right.$$
$$\left.-m\tau'\right]\varepsilon\left(\tau_b-\frac{z}{b}\right). \qquad (5\text{-}85)$$

The initial condition

$$T(z,x,y,0)=f(z,x,y). \qquad (5\text{-}86)$$

The boundary conditions

$$\frac{\partial T(b\tau_b, x, y, \tau')}{\partial z} = h_1[T_1 - T(b\tau_b, x, y, \tau')];$$

$$\frac{\partial T(-\infty, x, y, \tau')}{\partial z} = 0;$$

$$\frac{\partial T(z, L, y, \tau')}{\partial x} = h_2[T_2 - T(z, L, y, \tau')];$$

$$\frac{\partial T(z, x, D, \tau')}{\partial y} = h_2[T_2 - T(z, x, D, \tau')]; \tag{5-87}$$

$$\frac{\partial T(z, 0, y, \tau')}{\partial x} = \frac{\partial T(z, x, 0, \tau')}{\partial y} = 0.$$

Here τ' is the time after stopping of the process of erection or the time during the period of conservation; τ_b is the time during which the mass was erected; $f(z, x, y)$ is the initial temperature distribution, determined from the previous solution, for example (5-83) and (5-84), where $\tau = \tau_b$; $b\tau_b$ is the height of the concrete mass.

Unit function ε is equal to:

$$\varepsilon\left(\tau_b - \frac{z}{b}\right) = \begin{cases} 1 & \text{where } \tau_b - \frac{z}{b} < \tau_b, \\ 0 & \text{where } \tau_b - \frac{z}{b} > \tau_b. \end{cases}$$

Let us assume

$$\theta = T_2 - T_1.$$

and apply to (5-85)-(5-87) a double cosine transform.

We then have

$$\frac{\partial \bar{\bar{\vartheta}}}{\partial \tau'} = a\frac{\partial^2 \bar{\bar{\vartheta}}}{\partial z^2} - K_{pr} a \bar{\bar{\vartheta}} - \frac{q_0}{c\gamma} N_p M_r \exp\left[-m\left(\tau_b - \frac{z}{b}\right) - m\tau'\right] \varepsilon\left(\tau_b - \frac{z}{b}\right);$$

$$\bar{\bar{\vartheta}}(z, 0) = T_2 N_p M_r - \bar{\bar{f}}_{pr}(z);$$

$$\frac{\partial \bar{\bar{\vartheta}}(b\tau_b, \tau')}{\partial z} = h_1[(T_2 - T_1) N_p M_r - \bar{\bar{\vartheta}}(b\tau_b, \tau')];$$

$$\frac{\partial \bar{\bar{\vartheta}}(-\infty, \tau')}{\partial z} = 0; \; \bar{\bar{\vartheta}}(-\infty, \tau') \neq \infty,$$

where

$$\overline{\overline{f}}_{pr}(z)=\int_0^L\int_0^D f(z,x,y)\cos\mu_p\frac{x}{L}\cos\varkappa_r\frac{y}{D}\,dx\,dy;$$

$$K_{pr}=\frac{\mu_p^2}{L^2}+\frac{\varkappa_r^2}{D^2}.$$

The substitution

$$\overline{\overline{\vartheta}}=(T_2-T_1)N_pM_r-\overline{\overline{\vartheta}}$$

and the replacement of variables

$$\xi=\tau_b-\frac{z}{b};\ \tau'=t'$$

give

$$\frac{\partial\overline{\overline{\vartheta}}}{\partial t'}=\frac{a}{b^2}\frac{\partial^2\overline{\overline{\vartheta}}}{\partial\xi^2}-K_{pr}a\overline{\overline{\vartheta}}+aK_{pr}(T_2-T_1)N_pM_r+$$

$$+\frac{q_0}{c\gamma}N_pM_r\exp[-m\xi-mt']\,\varepsilon(\xi);$$

$$\overline{\overline{\vartheta}}(\xi,0)=\overline{\overline{f}}_{pr}(\xi)-T_1N_pM_r;$$

$$\frac{\partial\overline{\overline{\vartheta}}(0,t')}{\partial\xi}=bh\overline{\overline{\vartheta}}(0,t');$$

$$\frac{\partial\overline{\overline{\vartheta}}(\infty,t')}{\partial\xi}=0;\ \overline{\overline{\vartheta}}(\infty,t')\neq\infty.$$

The transform

$$\overline{\overline{\vartheta}}=u\exp[-aK_{pr}t']$$

brings the problem to the form

$$\frac{\partial u}{\partial t'}=\frac{a}{b^2}\frac{\partial^2u}{\partial\xi^2}+F(\xi,t');$$

$$u(\xi,0)=\Psi(\xi);$$

$$\frac{\partial u(0,t')}{\partial\xi}=mh_1u(0,t');$$

$$\frac{\partial u(\infty,t')}{\partial\xi}=0;\ u(\infty,t')\neq\infty, \qquad (5\text{-}88)$$

where

$$F(\xi, t') = aK_{pr}(T_2 - T_1) N_p M_r \exp[aK_{pr}t'] + \frac{q_0}{c\gamma} N_p M_r \times$$
$$\times \exp[-m\xi + (aK_{pr} - m) t'];$$
$$\Psi(\xi) = \overline{\overline{f}}_{pr}(\xi) - T_1 N_p M_r;$$

$$\varepsilon(\xi) = \begin{cases} 1 & \text{where } \xi < \tau_b, \\ 0 & \text{where } \xi > \tau_b. \end{cases}$$

The solution of problem (5-88) is determined like the solution of problem (5-82).

As a result we produce

$$T = T_2 + \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} A_p B_r \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} W_{pr}(\xi, t'), \tag{5-89}$$

where

$$W_{pr}(\xi, t') = \frac{e^{-K_{pr}at'}}{N_p M_r} \frac{b}{2\sqrt{\pi a t'}} \int_0^\infty \overline{\overline{f}}_{pr}(\eta) \left(\exp\left[-\frac{b^2(\xi - \eta)^2}{4at'} \right] + \right.$$
$$+ \exp\left[-\frac{b^2(\xi + \eta)^2}{4at'} \right] - \frac{2\sqrt{\pi a t'}}{b} \exp[h_1^2 at' + bh_1(\xi + \eta)] \times$$
$$\left. \times \operatorname{erfc}\left[h_1 \sqrt{at'} + \frac{b(\xi + \eta)}{2\sqrt{at'}} \right] \right) d\eta - T_2 \exp[-aK_{pr}t'] \times$$

$$\times \left(1 - \operatorname{erfc}\left[\frac{b\xi}{2\sqrt{at'}} \right] - \operatorname{erfc}\left[h_1 \sqrt{at'} + \frac{b\xi}{2\sqrt{at'}} \right] \right) +$$
$$+ (T_2 - T_1) \left\{ \frac{1}{2\sqrt{E_{pr}}} (h_1 - \sqrt{E_{pr}}) \exp[-(h_1 + \sqrt{E_{pr}}) b\xi] \times \right.$$
$$\times \operatorname{erfc}\left[-\sqrt{E_{pr}at'} + \frac{b\xi}{2\sqrt{at'}} \right] - \frac{1}{2\sqrt{E_{pr}}} (h_1 + \sqrt{E_{pr}}) \times$$
$$\times \exp[-(h_1 - \sqrt{E_{pr}}) b\xi] \operatorname{erfc}\left[\sqrt{E_{pr}at'} + \frac{b\xi}{2\sqrt{at'}} \right] - \frac{1}{2} \times$$
$$\times \exp[-b\xi\sqrt{K_{pr}}] \operatorname{erfc}\left[-\sqrt{K_{pr}at'} + \frac{b\xi}{2\sqrt{at'}} \right] - \frac{1}{2} \exp[b\xi\sqrt{K_{pr}}] \times$$
$$\left. \times \operatorname{erfc}\left[\sqrt{K_{pr}at'} + \frac{b\xi}{2\sqrt{at'}} \right] \right\} + \frac{q_0}{2c\gamma} \left(R_{pr} - \frac{m^2}{b^2} \right)^{-1} \times$$
$$\times \left\{ \exp\left[-\left(R_{pr} - \frac{m^2}{b^2} \right) at' - m\xi \right] \operatorname{erfc}\left[\frac{m}{b} \sqrt{at'} - \frac{b\xi}{2\sqrt{at'}} \right] - \right.$$

$$
\begin{aligned}
&-\exp\left[-\left(R_{pr}-\frac{m^2}{b^2}\right)at'-m\xi\right]\operatorname{erfc}\left[\frac{m}{b}\sqrt{at'}+\frac{b(\tau_b-\xi)}{2\sqrt{at'}}\right]-\\
&-\frac{bh_1+m}{bh_1-m}\exp\left[-\left(R_{pr}-\frac{m^2}{b^2}\right)at'+m\xi\right]\operatorname{erfc}\left[\frac{m}{b}\sqrt{at'}+\right.\\
&\left.+\frac{b\xi}{2\sqrt{at'}}\right]+\frac{bh_1+m}{bh_1-m}\exp\left[-\left(R_{pr}-\frac{m^2}{b^2}\right)at'+m\xi\right]\times\\
&\times\operatorname{erfc}\left[\frac{m}{b}\sqrt{at'}+\frac{b(\tau_b+\xi)}{2\sqrt{at'}}\right]-\frac{2bh_1}{bh_1-m}\exp\left[(h_1-m)\tau_b-\right.\\
&\left.-\left(R_{pr}-\frac{m^2}{b^2}\right)at'+bh_1\xi\right]\operatorname{erfc}\left[h_1\sqrt{at'}+\frac{b(\tau_b+\xi)}{2\sqrt{at'}}\right]+\\
&+\frac{1}{2\sqrt{R_{pr}}}(\sqrt{R_{pr}}-h_1)^{-1}(\sqrt{R_{pr}}+h_1)\left(\sqrt{R_{pr}}+\frac{m}{b}\right)\times\\
&\times\exp[-m\tau_b+b(\tau_b+\xi)\sqrt{R_{pr}}]\operatorname{erfc}\left[\sqrt{R_{pr}at'}+\frac{b(\tau_b+\xi)}{2\sqrt{at'}}\right]+\\
&+\frac{1}{2\sqrt{R_{pr}}}(\sqrt{R_{pr}}+h_1)^{-1}(\sqrt{R_{pr}}-h_1)\left(\sqrt{R_{pr}}-\frac{m}{b}\right)\times\\
&\times\exp[m\tau_b-b(\tau_b+\xi)\sqrt{R_{pr}}]\operatorname{erfc}\left[-\sqrt{R_{pr}at'}+\frac{b(\tau_b+\xi)}{2\sqrt{at'}}\right]+\\
&+\frac{1}{2\sqrt{R_{pr}}}\left(\sqrt{R_{pr}}+\frac{m}{b}\right)\exp[-m\tau_b+b(\tau_b-\xi)\sqrt{R_{pr}}]\operatorname{erfc}\left[\sqrt{R_{pr}at'}+\right.\\
&\left.+\frac{b(\tau_b-\xi)}{2\sqrt{at'}}\right]-\frac{1}{2\sqrt{R_{pr}}}\left(\sqrt{R_{pr}}-\frac{m}{b}\right)\exp[m\tau_b-b(\tau_b-\xi)\sqrt{R_{pr}}]\times\\
&\times\operatorname{erfc}\left[\sqrt{R_{pr}at'}-\frac{b(\tau_b-\xi)}{2\sqrt{at'}}\right]-\varepsilon\left(2\exp[-m\xi]-\right.\\
&-\frac{1}{\sqrt{R_{pr}}}\left(\sqrt{R_{pr}}-\frac{m}{b}\right)\exp[m\tau_b-b(\tau_b+\xi)\sqrt{R_{pr}}]-
\end{aligned}
$$

$$
\begin{aligned}
&\left.-\frac{1}{\sqrt{R_{pr}}}\left(\sqrt{R_{pr}}+\frac{m}{b}\right)\exp[-m\tau_b+b(\tau_b+\xi)\sqrt{R_{pr}}]\right)\Big\}-\\
&-\frac{q_0}{ac\gamma}\left\{\frac{bh_1}{(bh_1-m)}(R_{pr}-h_1^2)^{-1}\exp\left[-\left(R_{pr}-\frac{b^2}{4a^2}\right)at'+\right.\right.\\
&\left.+bh_1\xi\right]\operatorname{erfc}\left[h_1\sqrt{at'}+\frac{b\xi}{2\sqrt{at'}}\right]+\frac{bh_1+m}{2}(\sqrt{R_{pr}}+h_1)^{-1}\times\\
&\times\left(R_{pr}-\frac{m^2}{b^2}\right)^{-1}\exp[-b\xi\sqrt{R_{pr}}]\operatorname{erfc}\left[-\sqrt{R_{pr}at'}+\frac{b\xi}{2\sqrt{at'}}\right]+\\
&+\frac{bh_1+m}{2}(\sqrt{R_{pr}}-h_1)^{-1}\left(R_{pr}-\frac{m^2}{b^2}\right)^{-1}\times\\
&\left.\times\exp[b\xi\sqrt{R_{pr}}]\operatorname{erfc}\left[\sqrt{R_{pr}at'}+\frac{b\xi}{2\sqrt{at'}}\right]\right\};\\
&E_{pr}=K_{pr}+h_1^2; \qquad\qquad (5\text{-}90)\\
&R_{pr}=K_{pr}-\frac{m}{a}.
\end{aligned}
$$

Two-dimensional (planar) temperature field.

Before erection. The differential equation

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{q_0}{c\gamma} \exp\left[-m\left(\tau - \frac{z}{b}\right)\right] \times$$
$$\times \varepsilon\left(\tau - \frac{z}{b}\right) - H(T - T_3). \qquad (5\text{-}91)$$

The boundary conditions of the problem are similar to (5-80) and (5-81), but it should be kept in mind that all functions depend only on (z, x).

As before, the term $H(T - T_3)$ is introduced to the differential equation in order to consider heat losses from the third (OY axis) dimension.

The solution

$$T = T_3 + \sum_{p=1}^{\infty} A_p \cos \mu_p \frac{x}{L} [V_{pH}(\xi, t) + U_{pH}(\xi, t)], \qquad (5\text{-}92)$$

where $V_{pH}(\xi, t)$ is the same as in formula (5-84), but S_{pr} should be replaced by

$$S_{pH} = \frac{\mu_p^2}{L^2} + \frac{H}{a} + \frac{b^2}{4a^2};$$
$$U_{pH}(\xi, t) = \frac{HT_3}{a}\left(S_{pH} - \frac{b^2}{4a^2}\right)^{-1} \left\{\frac{h}{\left(h - \frac{b^2}{2a}\right)}\left(S_{pH} - \frac{h^2}{b^2}\right)^{-1} \times\right.$$
$$\times\left(S_{pH} - \frac{b^2}{4a^2}\right) \exp\left[-\left(S_{pH} - \frac{b^2}{4a^2}\right) at + \left(h + \frac{b^2}{2a}\right)\xi\right] \times$$
$$\times \operatorname{erfc}\left[-\frac{h}{b}\sqrt{at} + \frac{b\xi}{2\sqrt{at}}\right] + \left(\frac{b}{4a\sqrt{S_{pH}}} - \frac{h}{2\sqrt{S_{pH}}}\right)\left(h - \frac{b^2}{2a}\right)^{-1} \times$$

$$\times\left(\sqrt{S_{pH}}+\frac{b}{2a}\right)+\frac{h}{2\sqrt{S_{pH}}}\left(\sqrt{S_{pH}}-\frac{h}{b}\right)^{-1}\left(h-\frac{b^2}{2a}\right)^{-1}\times$$
$$\times\left(S_{pH}-\frac{b^2}{4a^2}\right)\Big)\exp\left[\left(b\sqrt{S_{pH}}+\frac{b^2}{2a}\right)\xi\right]\operatorname{erfc}\left[\sqrt{S_{pH}at}+\right.$$
$$\left.+\frac{b\xi}{2\sqrt{at}}\right]-\left(\frac{b}{4a\sqrt{S_{pH}}}+\frac{h}{2\sqrt{S_{pH}}}\left(h-\frac{b^2}{2a}\right)^{-1}\left(\sqrt{S_{pH}}-\frac{b}{2a}\right)+\right.$$
$$+\frac{h}{2b\sqrt{S_{pH}}}\left(\sqrt{S_{pH}}+\frac{h}{b}\right)^{-1}\left(h-\frac{b^2}{2a}\right)^{-1}\left(S_{pH}-\frac{b^2}{4a^2}\right)\Big)\times$$
$$\times\exp\left[-\left(b\sqrt{S_{pH}}-\frac{b^2}{2a}\right)\xi\right]\operatorname{erfc}\left[-\sqrt{S_{pH}at}+\frac{b\xi}{2\sqrt{at}}\right]-$$
$$-\frac{1}{2}\exp\left[-\left(\sqrt{S_{pH}}-\frac{b^2}{4a^2}\right)at\right]\Big(\operatorname{erfc}\left[\frac{b}{2a}\sqrt{at}-\frac{b\xi}{2\sqrt{at}}\right]-$$
$$-\frac{h+\frac{b^2}{2a}}{h-\frac{b^2}{2a}}\exp\left[\frac{b^2}{a}\xi\right]\operatorname{erfc}\left[\frac{b}{2a}\sqrt{at}+\frac{b\xi}{2\sqrt{at}}\right]\Big)\Big\}. \tag{5-92'}$$

Period of conservation. The differential equation and boundary conditions of the problem are obvious from the above.

The solution

$$T=T_2+\sum_{p=1}^{\infty}A_p\cos\mu_p\frac{x}{L}\left[W_{pH}(\xi,\ t')+\Lambda_{pH}(\xi,\ t')\right]; \tag{5-93}$$

where $W_{pH}(\xi,\ t')$ is defined by an expression similar to (5-90), but K_{pr}, R_{pr} and E_{pr} should be replaced by

$$K_{pH}=\frac{\mu_p^2}{L^2}+\frac{H}{a};\ R_{pH}=K_{pH}-\frac{m}{a};\ E_{pH}=K_{pH}+h_1^2;$$
$$\Lambda_{pH}(\xi,\ t')=\frac{1}{aK_{pH}}\left\{\frac{h_1-\sqrt{E_{pH}}}{2\sqrt{K_{pH}}}\exp\left[-(h_1+\sqrt{E_{pH}})b\xi\right]\times\right.$$
$$\times\operatorname{erfc}\left[-\sqrt{E_{pH}at'}+\frac{b\xi}{2\sqrt{at'}}\right]-\frac{h_1+\sqrt{E_{pH}}}{2\sqrt{E_{pH}}}\times$$
$$\times\exp\left[-(h_1-\sqrt{E_{pH}})b\xi\right]\operatorname{erfc}\left[\sqrt{E_{pH}at'}+\frac{b\xi}{2\sqrt{at'}}\right]-$$
$$-\frac{1}{2}\exp\left[-b\xi\sqrt{K_{pH}}\right]\operatorname{erfc}\left[-\sqrt{K_{pH}at'}+\frac{b\xi}{2\sqrt{at'}}\right]-$$
$$\left.-\frac{1}{2}\exp\left[b\xi\sqrt{K_{pH}}\right]\operatorname{erfc}\left[\sqrt{K_{pH}at'}+\frac{b\xi}{2\sqrt{at'}}\right]\right\}. \tag{5-94}$$

One-dimensional temperature field.

Period of erection.

The differential equation

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial z^2} + \frac{q_0}{c\gamma} \exp\left[-m\left(\tau - \frac{z}{b}\right)\right] \varepsilon\left(\tau - \frac{z}{b}\right) - H(T - T_3).$$

The initial condition

$$T(z, 0) = T_0.$$

The boundary conditions

$$\frac{\partial T(b\tau, \tau)}{\partial z} = H_1[\Phi_1 - T(b\tau, \tau)]; \quad \frac{\partial T(-\infty, \tau)}{\partial z} = 0, \; T)(-\infty, \tau) \neq \infty.$$

The solution

$$T = V_H(\xi, t) + U_H(\xi, t).$$

The functions $V_H(\xi, t)$ and $U_H(\xi, t)$ are defined by expressions (5-84) and (5-92'), but we must substitute

$$S_{pr} = S_{pH} = \frac{H}{a} + \frac{b^2}{4a^2}; \; T_2 = 0.$$

Period of conservation.

The solution

$$T = W_H(\xi, t') + \Lambda_H(\xi, t),$$

where $W_H(\xi, t')$ and $\Lambda_H(\xi, t')$ are similar to (5-90) and (5-94), but

$$K_{pH} \to K_H = \frac{H}{a}; \; R_{pH} \to R_H = K_H - \frac{m}{a}; \; E_{pH} \to E_H = K_H + h_1^2.$$

Consideration of Dependence of Ambient Temperature on Time

Spatial temperature field.

In the previous solutions, we assumed that the ambient temperature on the side surface of a column T_2, as well as a certain equivalent ambient temperature on the horizontal surface Φ_1 were constant.

However "high" concrete blocks are erected over the course of three days, and therefore, the nonconstancy of the temperature has an influence on the results of calculation during the period of construction.

In the case when this effect cannot be ignored, the problem is solved, for example, as follows.

Suppose $\Phi(\tau)$ and $\phi(\tau)$ are the variable components of ambient temperature on the horizontal and side surfaces respectively.

The general solution of the problem can be represented as the sum

$$T = (T)_1 + (T)_2,$$

where $(T)_1$ is the solution produced in the previous paragraph, considering heat liberation in the concrete and the constant components of ambient temperature $T_2 = \psi(0)$ and $\Phi_1 = \phi(0)$; $(T)_2$ is the solution of the problem for a continuously growing mass without heat liberation with zero initial temperature of the base and variable component of ambient temperature $\psi(\tau)$ and $\phi(\tau)$.

It is not difficult to show that solution $(T)_2$ in this case will be

$$(T)_2 = \varphi(\tau) + \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} A_p B_r \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} Q_{pr}(\xi, t), \tag{5-95}$$

where

$$Q_{pr}(\xi, t) = \frac{aK_{pr}}{2} e^{\frac{b^2}{2a}\xi} \int_0^t \left[\psi(\tau) - \varphi(\tau) - \frac{\varphi'(\tau)}{aK_{pr}} \right] \times$$

$$\times \left[J_1(\xi, \tau) + J_2(\xi, \tau) - 2hJ_3(\xi, \tau) \right] d\tau;$$

$$J_1(\xi, \tau) = \exp \left[- S_{pr} a (t - \tau) - G \left(\xi - \frac{Ga}{b^2}(t - \tau) \right) \right] \times$$

$$\times\left(\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}-\frac{b\xi}{2\sqrt{a(t-\tau)}}\right]-\right.$$
$$\left.-\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}+\frac{b(t-\xi)}{2\sqrt{a(t-\tau)}}\right]\right);$$
$$J_2(\xi,\tau)=\exp\left[-S_{pr}a(t-\tau)+G\left(\xi+\frac{Ga}{b^2}(t-\tau)\right)\right]\times$$
$$\times\left(\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}+\frac{b\xi}{2\sqrt{a(t-\tau)}}\right]-\right.$$
$$\left.-\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}+\frac{b(t+\xi)}{2\sqrt{a(t-\tau)}}\right]\right);$$
$$J_3(\xi,\tau)=\frac{1}{h-G}\exp\left[-S_{pr}a(t-\tau)+\frac{h^2a}{b^2}(t-\tau)+h\xi\right]\times$$
$$\times\left(\exp[(h-G)t]\operatorname{erfc}\left[\frac{h}{b}\sqrt{a(t-\tau)}+\frac{b(t+\xi)}{2\sqrt{a(t-\tau)}}\right]-\right.$$
$$\left.-\operatorname{erfc}\left[\frac{h}{b}\sqrt{a(t-\tau)}+\frac{b\xi}{2\sqrt{a(t-\tau)}}\right]\right)-$$
$$-\frac{1}{h-G}\exp\left[-S_{pr}a(t-\tau)+\frac{G^2a}{b^2}(t-\tau)+G\tau\right]\times$$
$$\times\left(\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}+\frac{b(t+\xi)}{2\sqrt{a(t-\tau)}}\right]-\right.$$
$$\left.-\operatorname{erfc}\left[\frac{G}{b}\sqrt{a(t-\tau)}+\frac{b\xi}{2\sqrt{a(t-\tau)}}\right]\right).$$

Two-dimensional temperature field.

Suppose the heat losses from the third dimension, along the OY axis, are considered by the term $H[T - \chi(\tau)]$, where $\chi(\tau)$ is the variable component of ambient temperature. Then

$$(T)_2=\varphi(\tau)+\sum_{p=1}^{\infty}A_p\cos\mu_p\frac{x}{L}P_{pH}(\xi,t),$$

where

$$P_{pH}(\xi,t)=\frac{aK_{pH}}{2}e^{\frac{b^2}{2a}\xi}\int_0^t\left[\psi(\tau)-\varphi(\tau)-\frac{1}{aK_{pH}}(\varphi'(\tau)-H\chi(\tau)+\right.$$
$$\left.+H\varphi(\tau))\right][J_1(\xi,\tau)+J_2(\xi,\tau)-2hJ_3(\xi,\tau)]\,d\tau.$$

One-dimensional temperature field.

The solution

$$(T)_2 = \varphi(\tau) + \frac{1}{2} e^{\frac{b^2}{2a}\xi} \int_0^t [\varphi'(\tau) - H\chi(\tau) +$$

$$+ H\varphi(\tau)] [J_1(\xi, \tau) + J_2(\xi, \tau) - 2hJ_3(\xi, \tau)]\, d\tau.$$

Regularization of One-Dimensional Temperature Field of a Continuously Growing Concrete Mass

It follows from the solutions produced in the previous sections that with sufficient duration of the process, the one-dimensional temperature field of a continually growing concrete mass (ignoring heat exchange on the side surfaces) far from the base is described by the expression

$$T = \Phi_1 + \frac{q_0}{c\gamma\left(m + \frac{m^2 a}{b^2}\right)} \left(\frac{h_1 + \frac{b}{a} + \frac{m}{b}}{h_1 + \frac{b}{a}} - \exp\left[-m\left(\tau - \frac{z}{b}\right)\right]\right). \tag{5-96}$$

This quasistable solution can be produced directly from the differential equation of the problem.

For the one-dimensional temperature field of a continually growing concrete mass, the problem is formulated as follows:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial z^2} + \frac{q_0}{c\gamma}\exp\left[-m\left(\tau - \frac{z}{b}\right)\right]\varepsilon\left(\tau - \frac{z}{b}\right)$$

$$(-\infty < z < b\tau,\ \tau > 0);$$

$$T(z, 0) = T_0\ (-\infty < z \leqslant b\tau);$$

$$\frac{\partial T(b\tau, \tau)}{\partial z} = H_1[\Phi_1 - T(b\tau, \tau)];$$

$$\frac{\partial T(-\infty, \tau)}{\partial z} = 0;\ T(-\infty, \tau) \neq \infty. \tag{5-97}$$

Here, as before,

$$\Phi_1 = \frac{h_1 T_1 + \frac{b}{a} T_0}{h_1 + \frac{b}{a}}; \quad H_1 = h_1 + \frac{b}{a}.$$

Let us perform replacement of variables

$$\theta = \Phi_1 - T_1,$$

and then assume:

$$\xi = \tau - \frac{z}{b}, \quad \tau = t.$$

We produce

$$\frac{\partial \vartheta}{\partial \tau} = \frac{a}{b^2} \frac{\partial^2 \vartheta}{\partial \xi^2} - \frac{\partial \vartheta}{\partial \xi} - \frac{1}{c\gamma} q_0 e^{-m\xi} \varepsilon(\xi);$$

$$\vartheta(\xi, 0) = \Phi_1 - T_0;$$

$$\frac{\partial \vartheta(0, t)}{\partial \xi} = bH_1 \vartheta(0, t);$$

$$\frac{\partial \vartheta(\infty, t)}{\partial \xi} = 0; \quad \vartheta(\infty, t) \neq \infty.$$

It follows from this that for the area $\xi < t$ with sufficient duration of the process

$$\frac{a}{b^2} \frac{d^2 \vartheta}{d\xi^2} - \frac{d\vartheta}{d\xi} - \frac{1}{c\gamma} q_0 e^{-m\xi} = 0;$$

$$\frac{d\vartheta(0)}{d\xi} = bH_1(0);$$

$$\frac{d\vartheta(\infty)}{d\xi} = 0; \quad \vartheta(\infty) \neq \infty. \tag{5-98}$$

As we can easily show, the solution to problem (5-98) is

$$\vartheta = -\frac{q_0}{c\gamma \left(m + \frac{m^2 a}{b^2}\right)} \left(\frac{m + bH_1}{bH_1} - e^{-m\xi}\right). \tag{5-99}$$

Solution (5-99) is identical to (5-96).

CHAPTER 6. METHODS OF CALCULATION OF TEMPERATURE FIELDS OF CONCRETE MASSES WITH COOLING PIPES

6-1. Statement of the Basic Problems

Cooling of concrete masses with water circulating through systems of pipes buried in the concrete is widely used in the construction of dams. Pipe, or forced, cooling has been used in the construction of the Hoover, Fontana, Shasta, Glen Canyon, Hungry Horse (USA), Warragamba (Australia), Grand Dixons (Switzerland), Bratsk, Krasnoyarsk (USSR) and other dams; it is presently in use in the construction of the Toktogul'skaya, Ingursk, Ust'-Ilimsk, Zeyskiy and other dams.

Pipe cooling is usually performed in two stages.

The first stage is performed immediately after pouring of the concrete. Its purpose is to prevent a sharp rise in temperature of the concrete due to hydration of the cement, thus reducing the difference between the maximum temperature of freshly poured concrete and the final stable temperature of the body of the structure. The duration of the first stage is 2 or 3 months or more.

The cooling pipes are shifted to the second stage after heat liberation in the concrete is practically completed. The primary purpose of this stage is to accelerate the transition to the temperature mode of the structure allowing sealing of the mass into a single unit as rapidly and reliably as possible. The sealing temperature is assumed fixed. The duration of the second stage is 2 to 3 months or more.

The cooling system consists of a number of horizontal coils consisting of pipes primarily 25 mm in diameter. The pipes are laid out horizontally at an interval of 0.75-1.90 m, 1.5 m in the USSR. The distance between coils in the vertical direction is generally the same as the height of a block (0.75-3.0 m). The spacing between pipes horizontally and vertically may differ in different zones of the dam.

The Fontana Dam (USA) was arbitrarily divided into 3 zones. Depending on the temperature at which the concrete was poured, the spacing between pipes in the first, bottom zone varied from 1.52 m both horizontally and vertically at a concrete mixture temperature of less than 13 C to 0.76 m at a concrete mixture temperature of 13-18 C. The concrete in the second zone was poured in blocks 1.52 m high, which defined the spacing between coils in the vertical direction, the spacing between pipes in the horizontal direction being 1.90 m. Finally, in the third zone, the horizontal and vertical spacings between pipes were 1.90 and 3.0 m respectively.

In the base zone of the dam of the Krasnoyarsk Power Plant, the following spacings were used: height of first, bottom block (or over old concrete) 0.7 m, of second block -- 1.0 m, of third block -- 1.3 m, of fourth and subsequent blocks -- 1.5 m. The pipes were laid in the horizontal construction seams between blocks at a spacing of 1 m for the first three blocks and 1.5 m for subsequent blocks. The spacing of 1.5 m, horizontally and vertically, was retained at the other levels of the dam. The installation of the cooling system for blocks 3 m and more high was performed at levels intermediate through the height of the blocks.

Pipe cooling can be used not only to perform those functions indicated in the description of the first and second stages; it is also an effective method of regulating the temperature in concrete masses which for some reason must be constructed without following technological instructions (elevated temperature of concrete mixture, variation from pouring schedule, etc.). In this case, it is desirable to switch on individual coils in the system for certain periods of time.

The cooling liquid is either river water or specially cooled water, brine being used sometimes for deep cooling.

The speed of movement of water in the pipes is assigned based on the condition of creation of a turbulent stream, which provides the greatest heat exchange. It is usually 0.5-1.0 m/s.

Selection of the length of a coil depends on the necessary rate of cooling of the concrete, the operating conditions of the system, the location of the coils, water consumption, etc. The coil length varies between 180 and 400 m.

The importance of regulation of the thermal mode of dams by means of pipe cooling requires that methods be developed for calculation, allowing us to plan cooling systems and determine their parameters, and also to predict the temperature of the mass.

The problems which arise in this case are quite complex. Therefore, different authors have suggested various methods for the description of the thermal state of masses cooled by pipes [14, 38, 51, 64, 65, 124, 152, 154, 169, 172, 173].

In this chapter, we will present the mathematical principles of the most widely used models -- the model of an unlimited hollow cylinder, the model of linear heat sources (sinks), and the model of temperature sources.

6-2. A Pipe in an Unlimited Space

A study is made of a temperature field in an unlimited space, caused by the cooling effect of a pipe of radius R, through which water flows at temperature T_ℓ. It is assumed that the walls of the pipe receive a temperature equal to the temperature of the cooling water, the heating of the water over the length of the pipe being ignored.

The mathematical formulation of the problem

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \quad (R < r < \infty,\ \tau > 0);$$
$$T(r, 0) = T_0 \quad (R \leqslant r < \infty);$$
$$T(R, \tau) = T_\ell,\ T(\infty, \tau) \neq \infty. \tag{6-1}$$

We assume

$$\vartheta = \frac{T - T_0}{T_\ell - T_0}.$$

Then

$$\frac{\partial \vartheta}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta}{\partial r}\right)\right] \quad (R < r < \infty,\ \tau > 0);$$
$$\vartheta(r, 0) = 0, \quad (R \leqslant r < \infty);$$
$$\vartheta(R, \tau) = 1,\ \vartheta(\infty, \tau \neq \infty).$$

The solution of this problem was presented in a book by G. Karslou and D. Yegera [54]. The result

$$\vartheta = \frac{T - T_0}{T_m - T_0} = 1 + \frac{2}{\pi}\int_0^\infty e^{-\zeta^2 \mathrm{Fo}} \frac{Y_0(\zeta) J_0\left(\zeta\frac{r}{R}\right) - J_0(\zeta) Y_0\left(\zeta\frac{r}{R}\right)}{J_0^2(\zeta) + Y_0^2(\zeta)} \frac{d\zeta}{\zeta}, \tag{6-2}$$

where Fo = $a\tau/R^2$ is the Fourier criterion.

The heat flux per unit length of pipe is

$$\eta = -\lambda \left.\frac{\partial T}{\partial r}\right|_{r=R} 2\pi R = \frac{8\lambda(T_m - T_0)}{\pi}\int_0^\infty e^{-\zeta^2 \mathrm{Fo}} \frac{d\zeta}{\zeta[J_0^2(\zeta) + Y_0^2(\zeta)]}. \tag{6-3}$$ [1]

---

[1] Since $T_\ell > T_0$, η has the negative sign, which corresponds to direction of heat flow toward the pipe, i.e., cooling of the mass.

For high values of the Fo criterion, the heat flux to the surface of the pipe is determined by the formula

$$\eta = 4\pi\lambda (T_{\text{ж}} - T_0) \left\{ \frac{1}{\ln(4\,\text{Fo}) - 2C} - \frac{C}{[\ln(4\,\text{Fo}) - 2C]^2} - \dots \right\}, \qquad (6\text{-}4)$$

where C = 0.57722... is Euler's constant.

Let us estimate the value of the criterion Fo = $a\tau/R^2$ for concrete masses with cooling pipes. We assume: pipe diameter 25.4 mm, temperature conductivity factor of concrete 0.004 m^2/hr.

We have: Fo ≃ 0.25 τ.

Consequently, 1-2 hr after feeding of water into the pipe, the Fo criterion takes on values allowing us to use formula (6-4).

6-3. Model of an Unlimited Hollow Cylinder

The hollow cylinder model was developed in studies conducted by the Bureau of Reclamation of the USA [172, 173], M. S. Lamkin [64, 65], A. G. Tkachev, G. N. Danilova, N. A. Buchko and N. N. Syrovtseva [14, 38, 124], G. I. Chilingarishvili and R. G. Kakauridze [51], R. Stucky and M. Derron [169] and others [154].

Suppose a system of pipes is laid in the mass, through which water circulates. Suppose the water temperature is constant in all pipes.

A transverse cross section of an area of the mass rather far from its boundaries is shown in Figure 6-1.

The axes of symmetry around each of the pipes can be used to separate its area of influence -- a hexagon if the pipes are laid out in checkerboard order (Figure 6-1a) or a rectangle if they are laid out in aligned rows (Figure 6-1b). Obviously, at the axes of symmetry the heat flux will be equal to 0.

We replaced these areas with a hollow cylinder, the internal radius of which is equal to the radius of the pipe, while the external radius is determined from the condition of equality of area to that of the hexagon or rectangle which the hollow cylinder replaces.

Then the external radius of the hollow cylinder with checkerboard and row placement of pipes will be determined by

$$R_2 = \sqrt{\frac{L_1 L_2}{\pi}},$$

where L_1 and L_2 are the spacings between pipes in the horizontal and vertical directions.

We note that a regular hexagon is produced where $L_1 = 1.1547\ L_2$.

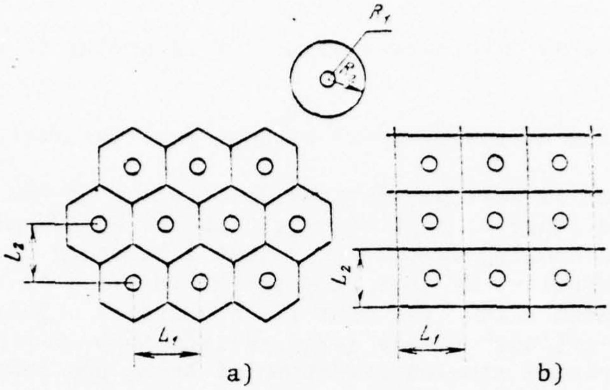


Figure 6-1. Diagram of Mass with Pipe Cooling. a, Checkerboard Placement of Pipes; b, Row Placement of Pipes

If we base our calculations on the equality of perimeters of these figures (as was suggested in [154], with row placement of pipes

$$R_2 = \frac{1}{\pi}(L_1 + L_2)$$

and with checkerboard placement

$$R_2 = \frac{1}{\pi}\sqrt{\frac{L_2^2}{9} + \frac{L_1^2}{4}} + 1,5L_2,$$

in the particular case of a regular hexagon

$$R_2 = \frac{3b}{\pi}$$ (b is one side of the hexagon).

Due to the high values of heat transfer coefficient with turbulent flow in the pipes, the pipe wall temperature can be assumed equal to the temperature of the water.

The outer surface of the hollow cylinder used in the model is considered to be heat insulated $\left(\frac{\partial T}{\partial r}\Big|_{r = R_2} = 0\right)$.

Let us assume that the water temperature does not change along the length of the pipe.

Let us now study several calculation plans corresponding to this assumption.

Temperature Field of a Hollow Cylinder without Heat Insulation

This calculation plan is suitable for the description of the thermal mode of a mass in the second stage of pipe cooling. As was noted earlier, the second stage of pipe cooling begins after heat liberation in the mass is practically over. Usually, by this time there are areas of significant size within the mass located rather far from its boundaries. Therefore, here the model of the hollow cylinder yields quite satisfactory results. This has been confirmed by special studies performed at Ariel Dam (USA) [173].

The formulation of the problem

$$\begin{gathered}\frac{\partial T}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \quad (R_1 < r < R_2,\ \tau > 0);\\ T(r,\ 0) = T_0 \quad (R_1 \leqslant r \leqslant R_2),\\ T(R_1,\ \tau) = T_{\text{ж}};\quad \frac{\partial T(R_2,\ \tau)}{\partial r} = 0.\end{gathered} \tag{6-5}$$

The solution (see Chapters 3 and 4)

$$T = T_{\text{ж}} + (T_0 - T_{\text{ж}})\sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{r}{R_1}\right) e^{-\mu_n^2 \mathrm{Fo}}, \tag{6-6}$$

where

$$U_0\left(\mu_n \frac{r}{R_1}\right) = Y_0(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_0(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right);$$

μ_n is the root of the characteristic equation

$$Y_0(\mu_n) J_1(k\mu_n) - J_0(\mu_n) Y_1(k\mu_n) = 0; \quad k = \frac{R_2}{R_1};$$

$$A_n = \frac{\pi J_0^2(k\mu_n)}{J_1^2(k\mu_n) - J_0^2(\mu_n)};$$

$$Fo = \frac{a\tau}{R_1^2}.$$

The maximum temperature is observed on the outer surface ($r = R_2$) of the cylinder, and is equal to:

$$T(R_2, \tau) = T_{max} = T_{ж} + (T_0 - T_{ж}) \sum_{n=1}^{\infty} A_n U_0(k\mu_n) e^{-\mu_n^2 Fo}.$$

But

$$U_0(k\mu_n) = Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n),$$

and considering the characteristic equation and known relationship

$$Y_0(z) J_1(z) - J_0(z) Y_1(z) = \frac{2}{\pi z}$$

we find:

$$U_0(k\mu_n) = -\frac{2}{\pi k\mu_n} \frac{J_0(\mu_n)}{J_1(k\mu_n)}.$$

From this

$$T_{max} = T_{ж} - \frac{2(T_0 - T_{ж})}{k\pi} \sum_{n=1}^{\infty} \frac{A_n J_0(\mu_n)}{\mu_n J_1(k\mu_n)} e^{-\mu_n^2 Fo}. \tag{6-7}$$

The mean integral temperature of the hollow cylinder $\overline{T}$ is determined by the expression

$$\bar{T} = \frac{\int\limits_{R_1}^{R_2} T 2\pi r\,dr}{\int\limits_{R_1}^{R_2} 2\pi r\,dr} = \frac{2\int\limits_{R_1}^{R_2} Tr\,dr}{(R_2^2 - R_1^2)}.$$

Consequently

$$\bar{T} = T_{ж} - \frac{4(T_0 - T_{ж})}{\pi(k^2 - 1)} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} e^{-\mu_n^2 \mathrm{Fo}}. \tag{6-8}$$

We consider here that

$$\int\limits_{R_1}^{R_2} rU_0\left(\mu_n \frac{r}{R_1}\right) dr = \frac{R_1}{\mu_n}\left[R_2 U_1(k\mu_n) - R_1 U_1(\mu_n)\right];$$

$$U_1(k\mu_n) = 0;\ U_1(\mu_n) = \frac{2}{\pi\mu_n}.$$

The heat flux per unit pipe length is

$$q = -\lambda \left.\frac{\partial T}{\partial r}\right|_{r=R_1} 2\pi R_1 = \pi(R_2^2 - R_1^2)\,c\gamma\,\frac{d\bar{T}}{d\tau} =$$
$$= 4\lambda(T_0 - T_{ж}) \sum_{n=1}^{\infty} A_n e^{-\mu_n^2 \mathrm{Fo}}. \tag{6-9}$$

Temperature Field of a Hollow Cylinder with Heat Liberation Dependent Only on Time

This calculation plan is used in determining the temperature field of a concrete mass in the first stage of pipe cooling. Since the cooling system operates in the first stage for the first 10 or 15 days after pouring of the concrete, the areas of the mass with pipes are basically located near the boundary. The symmetry of the temperature field, at least in the vertical direction, which is necessary in order to divide the calculation area into hexagons or rectangles, is not observed, and the model of the hollow

cylinder is but a rough approximation of reality. Nevertheless, it may be useful in estimating the effect of pipe cooling for areas sufficiently far from the boundary, etc.

In this case the problem is formulated as

$$\frac{\partial T}{\partial \tau}=a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]+\frac{1}{c\gamma}q(\tau)\ (R_1<r<R_2,\ \tau>0);$$
$$T(r,\ 0)=T_0\ (R_1\leq r\leq R_2);$$
$$T(R_1,\ \tau)=T_{\text{п}};\ \frac{\partial T(R_2,\ \tau)}{\partial r}=0. \tag{6-10}$$

We assume

$$T=\theta_1+\theta_2,$$

where θ_1 satisfies the homogeneous differential equation of heat conductivity and the edge conditions

$$\theta_1(r,\ 0)=T_0;$$
$$\theta_1(R_1,\ \tau)=T_{\text{п}};\ \frac{\partial\theta_1(R_2,\ \tau)}{\partial r}=0,$$

θ_2 satisfies the heterogeneous equation from (6-10) and the homogeneous edge conditions. Since the solution of the problem for function θ_1 was presented earlier, let us discuss the determination of θ_2, representing it now as T.

Suppose the heat liberation intensity function depends exponentially on time

$$q(\tau)=q_0e^{-m\tau}.$$

Then

$$T(r,\ \tau)=\frac{q_0R_1^2}{\lambda}\sum_{n=1}^{\infty}\frac{A_n}{\mu_n^2-m^{*2}}U_0\left(\mu_n\frac{r}{R_1}\right)\left(e^{-m\tau}-e^{-\mu_n^2\mathrm{Fo}}\right) \tag{6-11'}$$

or

$$T(r, \tau) = \frac{q_0 R_1^2}{\lambda} \left[w(r) e^{-m\tau} - \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 - m^{*2}} U_0\left(\mu_n \frac{r}{R_1}\right) e^{-\mu_n^2 \mathrm{Fo}} \right], \tag{6-11}$$

where

$$w(r) = \frac{1}{m^{*2}} \left[\frac{Y_1(km^*) J_0\left(m^* \frac{r}{R_1}\right) - J_1(km^*) Y_0\left(m^* \frac{r}{R_1}\right)}{J_0(m^*) Y_1(km^*) - Y_0(m^*) J_1(km^*)} - 1 \right];$$

$$m^{*2} = \frac{mR_1^2}{a}; \; k = \frac{R_2}{R_1}.$$

The remaining symbols are the same as before.

The maximum temperature

$$T_{max} = \frac{q_0 R_1^2}{\lambda} \left[w_m e^{-m\tau} + \frac{2}{k\pi} \sum_{n=1}^{\infty} \frac{A_n J_0(\mu_n)}{\mu_n (\mu_n^2 - m^{*2}) J_1(k\mu_n)} e^{-\mu_n^2 \mathrm{Fo}} \right], \tag{6-12}$$

where

$$w_m = \frac{1}{m^{*2}} \left\{ \frac{2}{km^*\pi} [Y_0(m^*) J_1(km^*) - J_0(m^*) Y_1(km^*)]^{-1} - 1 \right\}.$$

The mean integral temperature

$$\overline{T} = \frac{4 q_0 R_1^2}{\pi (k^2 - 1) \lambda} \left[\overline{w} e^{-m\tau} + \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 (\mu_n^2 - m^{*2})} e^{-\mu_n^2 \mathrm{Fo}} \right]. \tag{6-13}$$

where

$$\overline{w} = \frac{\pi}{4m^{*2}} \left\{ \frac{Y_1(m^*) J_1(km^*) - J_1(m^*) Y_1(km^*)}{m^* [J_0(m^*) Y_1(km^*) - Y_0(m^*) J_1(km^*)]} - 1 \right\}.$$

The heat flux per unit pipe length

$$\eta = 4q_0R_1^2\left[w_1 e^{-m\tau} - \sum_{n=1}^{\infty}\frac{A_n}{\mu_n^2 - m^{*2}} e^{-\mu_n^2 F_0}\right],\tag{6-14'}$$

where

$$w_1 = \frac{\pi}{2m^*}\frac{Y_1(km^*)J_1(m^*) - J_1(km^*)Y_1(m^*)}{Y_1(km^*)J_0(m^*) - J_1(km^*)Y_0(m^*)}.\tag{6-14}$$

If the heat liberation intensity function is an arbitrary function of time $q(\tau)$, then, approximating it by the piecewise-continuous function

$$q = q_\nu e^{-m_\nu \tau}\ (\nu = 1,\ 2, \ldots, s),$$

where q_ν, m_ν are constants, defined over $(\tau_{\nu-1}, \tau_\nu)$, we find

$$T(r, \tau) = \frac{R_1^2}{\lambda}\sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{r}{R_1}\right)\exp\left[-\mu_n^2 \frac{a\tau_s}{R_1^2}\right]\times$$
$$\times\left\{\sum_{\nu=1}^{s}\frac{q_\nu}{\mu_n^2 - m_\nu^{*2}}\left(\exp\left[\left(\mu_n^2\frac{a}{R_1^2} - m_\nu\right)\tau_\nu\right] -\right.\right.$$
$$\left.\left. - \exp\left[\left(\mu_n^2\frac{a}{R_1^2} - m_\nu\right)\tau_{\nu-1}\right]\right)\right\};\ m_\nu^{*2} = \frac{m_\nu R_1^2}{a}.\tag{6-15}$$

In the case of approximation of $q(\tau)$ by the sum

$$q = \sum_{\nu=1}^{i} q_\nu e^{-\nu m\tau}$$

the temperature function $T(r, \tau)$ will be equal to the sum s of terms of the form (6-11).

Temperature Field of a Hollow Cylinder with Heat Liberation Dependent on Temperature and Time

The formulation of the problem

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \frac{1}{c\gamma}q(\tau, T);$$
$$T(r, 0) = T_0;$$
$$T(R_1, \tau) = T_{ж}; \quad \frac{\partial T(R_2, \tau)}{\partial r} = 0. \tag{6-16}$$

Here $q(\tau, T)$ is the heat liberation intensity function.

Let us discuss the two representations $q(\tau, T)$ in the form of a generalized function of heat liberation intensity and in the form of the function of I. D. Zaporozhets.

1. The generalized heat liberation intensity function

$$q(\tau, T) = q_\nu(d_\nu + b_\nu T)e^{-m_\nu \tau} \quad (\nu = 1, 2, \ldots, s),$$

where q_ν, d_ν, b_ν, m_ν are constants, defined in $(\tau_{\nu-1}, \tau_\nu)$.

The temperature function $T(r, \tau)$ is equal to (see Chapter 4)

$$T(r, \tau) = T_{ж} - \sum_{n=1}^{\infty} A_n U_0\left(\mu_n \frac{r}{R_1}\right)\exp\left[-\mu_n^2 \frac{a\tau_s}{R_1^2}\right]\exp\left[\sum_{\nu=1}^{s} c_\nu\right]\times$$
$$\times\left(B_n - \sum_{\nu=1}^{s} L_\nu \Psi_{\nu n}\exp\left[-\sum_{\nu'=1}^{\nu} c_{\nu'}\right]\right). \tag{6-17}$$

where

$$B_n = T_{ж} - T_0;$$

$$L_\nu = \frac{1}{c\gamma}(q_\nu d_\nu + q_\nu b_\nu T_{ж});$$

$$c_\nu = \frac{q_\nu b_\nu}{m_\nu c\gamma}(\exp[-m_\nu \tau_{\nu-1}] - \exp[-m_\nu \tau_\nu]);$$

$$\Psi_{\nu n} = \exp\left[-\frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \tau_\nu}\right] \int_{\tau_{\nu-1}}^{\tau_\nu} \exp\left[\mu_n^2 \frac{a\zeta}{R_1^2} - m_\nu \zeta + \right.$$

$$\left. + \frac{q_\nu b_\nu}{m_\nu c\gamma} e^{-m_\nu \zeta}\right] d\zeta.$$

The formulas for calculation of the maximum temperature T_{max}, mean temperture $\overline{T}$ and heat flux per unit pipe length η are obvious from the above.

2. The heat liberation intensity function of I. D. Zaporozhets

$$q(\tau, T) = q_0 2^{\frac{T-20}{\theta}} \left[1 + A_{20} \int_0^\tau 2^{\frac{T-20}{\theta}} d\tau\right]^{-\frac{m}{m-1}}, \tag{6-18}$$

where

$$q_0 = \frac{Q_{max} A_{20}}{m-1}.$$

The solution of problem (6-16) is produced by the method of finite differences. Let us introduce the space-time grid $w_{h\ell} = \{r_1 = R_1 + ih;\ \tau_k = k\ell;\ i = 0, 1, \ldots, n;\ k = 0, 1, \ldots, s\}$ and the grid function $T_{i,k}$.

The derivatives included in the differential equation from (6-16) can be replaced by finite-difference relationships as follows:

$$\left(\frac{\partial T}{\partial \tau}\right)_i \sim \frac{T_{i,k+1} - T_{i,k}}{l};$$

$$\left(\frac{\partial^2 T}{\partial r^2}\right)_p \sim \frac{T_{i+1,p} - 2T_{i,p} + T_{i-1,p}}{h^2} \ (p = k,\ k+1);$$

$$\left(\frac{\partial T}{\partial r}\right)_p \sim \frac{T_{i+1,p} - T_{i-1,p}}{2h}.$$

Then the differential equation from (6-16) is approximated by a set of finite difference plans such as

$$\frac{T_{i,k+1}-T_{i,k}}{l}=$$
$$=\sigma a\frac{\left(r_i-\frac{h}{2}\right)T_{i-1,k+1}-2r_iT_{i,k+1}+\left(r_i-\frac{h}{2}\right)T_{i+1,k+1}}{r_ih^2}+$$
$$+(1-\sigma)a\frac{\left(r_i-\frac{h}{2}\right)T_{i-1,k}-2r_iT_{i,k}+\left(r_i-\frac{h}{2}\right)T_{i+1,k}}{r_ih^2}, \qquad (6\text{-}19)$$

where σ is a parameter ($0 \leq \sigma \leq 1$).

Let us utilize the predictor-corrector plan (see § 4-2).

First stage. We select the spacing 0.5 ℓ and substitute in formula (6-19) $\sigma = 1$.

We have:

$$\frac{1}{2}M\left(1-\frac{h}{2r_i}\right)T_{i-1,k+1/2}-(1+M)T_{i,k+1/2}+$$
$$+\frac{1}{2}M\left(1-\frac{h}{2r_i}\right)T_{i+1,k+1/2}=-\vartheta^{(k)} \quad (1\leqslant i\leqslant n-1,\ 0<k\leqslant s-1);$$
$$T_{i,0}=T_0 \quad (0\leqslant i\leqslant n);$$
$$T_{0,k+1/2}=T_{\text{ж}};$$
$$T_{n,k+1/2}=\varkappa_2T_{n-1,k+1/2}+\vartheta_2^{(k)}. \qquad (6\text{-}20)$$

where

$$\vartheta^{(k)}=T_{i,k}+\frac{q_0l}{2c\gamma}2^{\frac{T_{i,k}-20}{\epsilon}}\left[1+A_{20}l\sum_{p=0}^{k}2^{\frac{T_{i,p}-20}{\epsilon}}\right]^{-\frac{m}{m-1}};$$
$$\varkappa_2=\frac{M}{1+M};\quad M=\frac{al}{2h^2};$$
$$\vartheta_2^{(k)}=\varkappa_2\left\{T_{n-1,k}+\frac{1-M}{M}T_{n,k}+\frac{q_0l}{2Mc\gamma}2^{\frac{T_{n,k}-20}{\epsilon}}\times\right.$$
$$\times\left[1+A_{20}l\sum_{p=0}^{k}2^{\frac{T_{n,p}-20}{\epsilon}}\right]^{-\frac{m}{m-1}}. \qquad (6\text{-}21)$$

In order to solve equation system (6-20) with a three-diagonal matrix, we can utilize the run-through method described in § 4-2. This gives us an intermediate value of temperature $T_{i,k+1/2}$.

Second stage. Let us assume in formula (6-19) $\sigma = 1/2$ and use the moment in time $\tau_{k+1/2}$ for temperature in expression (6-18).

We produce the difference problem

$$\begin{gathered}\frac{1}{2} M\left(1-\frac{h}{2r_1}\right) T_{i-1,k+1}-(1+M) T_{i,k+1}+\\ +\frac{1}{2} M\left(1-\frac{h}{2r_1}\right) T_{i+1,k+1}=-\vartheta^{(k)},\ (1 \leqslant i \leqslant n-1,\ 0<k \leqslant s-1);\\ T_{i,0}=T_0 \quad (0 \leqslant i \leqslant n);\\ T_{0,k+1}=T_{k};\\ T_{n,k+1}=\varkappa_2 T_{n-1,k+1}+\vartheta_2^{(k)},\end{gathered} \tag{6-22}$$

where

$$\begin{gathered}\vartheta^{(k)}=\frac{1}{2} M\left(1-\frac{h}{2r_1}\right)\left(T_{i-1,k}+T_{i+1,k}\right)+\\ +(1-M) T_{i,k}+\frac{q_0 l}{c\gamma} 2^{\frac{T_{i,k+1/2}-20}{\varepsilon}}\left[1+A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{i,p}-20}{\varepsilon}}+\right.\\ \left.+A_{20} \frac{l}{2} 2^{\frac{T_{i,k+1/2}-20}{\varepsilon}}\right]^{-\frac{m}{m-1}};\end{gathered} \tag{6-23}$$

$$\begin{gathered}\vartheta_2^{(k)}=\varkappa_2\left\{T_{n-1,k}+\frac{1-M}{M} T_{n,k}+\right.\\ +\frac{q_0 l}{c\gamma} 2^{\frac{T_{n,k+1/2}-20}{\varepsilon}}\left[1+A_{20} l \sum_{p=0}^{k} 2^{\frac{T_{n,p}-20}{\varepsilon}}+A_{20} \frac{l}{2} 2^{\frac{T_{n,k+1/2}-20}{\varepsilon}}\right]^{-\frac{m}{m-1}}.\end{gathered} \tag{6-24}$$

Solving equation system (6-22) by the run-through method, in order to determine the run-through coefficients ν_{i+1} and ε_{i+1} by the "direct run" method and the temperature function $T_{i,k+1}$ by the "reverse run" method, we utilize the algorithm

$$\nu_{i+1} = \frac{0{,}5M(1 - h/2r_i)}{(1+M) - 0{,}5M(1 - h/2r_i)\nu_i};$$

$$\varepsilon_{i+1} = \frac{0{,}5M(1 - h/2r_i)\varepsilon_i + \theta^{(k)}}{(1+M) - 0{,}5M(1 - h/2r_i)\nu_i} \quad (i = 1, 2, \ldots, n-1);$$

$$\nu_1 = 0, \ \varepsilon_1 = 1;$$

$$T_{i,k+1} = \nu_{i+1}T_{i+1,k+1} + \varepsilon_{i+1} \ (i = 0, 1, \ldots, n-1);$$

$$T_{n,k+1} = \frac{\theta_2^{(k)} + \varkappa_2\varepsilon_n}{1 - \varkappa_2\nu_n}. \tag{6-25}$$

It is not difficult to see that the run-through conditions in our case reduced to

$$1 - \frac{h}{2r_i} > 0 \ (i = 1, 2, \ldots, n), \tag{6-26}$$

are always fulfilled (since $r_i = R_1 + ih$).

Calculation of Heating of Water in Pipes

As it runs through the pipes laid in the concrete, the water is warmed. This leads to different rates of cooling of the concrete along the length of each pipe which, obviously, should be considered in calculation formulas more precise than those presented above.

As before, we are studying a model of an unlimited hollow cylinder with heat insulation of the outer surface, the temperature of the inside surface of the cylinder being equal to the temperature of the water flowing along axis OZ. Heat liberation occurs in the concrete. The initial temperature of the concrete is constant at T_0, the temperature of the water in a certain cross section of the pipe (for which we assume $Z = 0$) is equal to θ_0. We assume that heat conductivity heat is not transmitted in the water but, due to the good mixing (turbulent flow), the temperature of the water is identical through the cross section of the pipe.

The problem is then formulated as follows.

The heat conductivity equation in the concrete

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \frac{1}{c_6\gamma_6}q(\tau, T)\ (R_1 < r < R_2,\ z > 0,\ \tau > 0). \tag{6-27}$$

The boundary conditions for it are

$$\frac{\partial\theta(R_1, z, \tau)}{\partial\tau} + u_z \frac{\partial\theta(R_1, z, \tau)}{\partial z} = \frac{\lambda_b}{M_l c_l} \frac{\partial T(R_1, z, \tau)}{\partial r};$$

$$T(R_1, z, \tau) = \theta(R_1, z, \tau). \tag{6-28}$$

The initial conditions

$$T(r, z, 0) = T_0; \; \theta(z, 0) = \theta(0, \tau) = \theta_0. \tag{6-29}$$

Here $T(r, z, \tau)$ is the temperature of the concrete; $\theta(z, \tau)$ is the temperature of the water; u_z is the speed of the stream of water (constant); $M_\ell = \gamma_\ell(R_1/2)$ is the mass of water per unit surface of pipe; the subscript "b" relates to the concrete, the subscript "ℓ" -- to the water; the remaining symbols are obvious.

The solution of problem (6-27)-(6-29) represents considerable difficulty. Therefore, it is usually assumed that the heat conductivity factor of the solid λ_b is finite (though $\lambda_b \neq 0$) in the direction perpendicular to the motion, i.e., in direction r, and equal to 0 in the direction of the motion (along z). In the case in question, this approach is justified, since the heating of the water in the pipes is not great.

The simplified heat conductivity equation in the concrete is then

$$\frac{\partial T}{\partial\tau} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \frac{1}{c\gamma} q(\tau, T).$$

The boundary and initial conditions remain unchanged [see (6-28) and (6-29)].

This task can be solved by means of a Laplace transform. However, inversion of the mappings of $\overline{T}$ and $\overline{\theta}$ thus produced is quite difficult. It is simpler to use the method suggested by the U.S. Bureau of Reclamation [173], tested in experimental studies of the thermal mode of concrete dams with cooling pipes [such as Ariel and Hoover (USA), etc.].

This method is also based on the assumption that the heat conductivity of the concrete is finite (but not 0) in the r direction and equal to 0 along the direction of the stream. Since the time of passage of water through the coil is slight in comparison to the process of cooling of the concrete,

it is assumed that the water passes through the coil instantly.

The American scientists [172, 173] have developed a method of calculation as applicable to concrete masses in which heat liberation is absent (i.e., for the second stage of cooling); further improvement and development for the case of concrete masses with internal heat liberation (first stage of cooling) was given to this method in the studies of A. G. Tkachev, G. N. Danilova, N. A. Buchko and N. N. Syrovtseva [14, 38, 41, 124], Chzhu-Bo-Fan [154] and others.

In works by the U.S. Bureau of Reclamation [172] and in a recently published book by N. A. Buchko and G. N. Danilova [14], a detailed description is presented of a method based on the model of the unlimited hollow cylinder for calculation of the heating of the water in pipes and the mean concrete temperature. We will therefore now present the basic calculation dependences of this method.

1. Temperature calculations for the second stage of pipe cooling. Heat liberation in the concrete is 0, the initial temperature of the concrete is T_{b0}, the temperature of water at the intake to the coil is $T_{\ell 0}$. We introduce the variables X_1, Y_1 and Z_1, equal to

$$X_1 = \frac{T_{bL} - T_{\ell 0}}{T_{b0} - T_{\ell 0}}; \quad Y_1 = \frac{T_{\ell L} - T_{\ell 0}}{T_{b0} - T_{\ell 0}}; \quad Z_1 = \frac{\overline{T}_{bL} - T_{\ell 0}}{T_{b0} - T_{\ell 0}}.$$

Here T_{bL} is the mean temperature of the concrete in the cross section located at distance L from the inlet; $\overline{T}_{bL}$ is the mean temperature of the concrete cylinder of length L; $T_{\ell L}$ is the water temperature at distance L from the intake.

Variable Y_1 (τ, L) is determined by the integral equation

$$Y_1 = \int_0^{\xi} \int_0^{\mathrm{Fo}_\tau} R\,[\mathrm{Fo}_{\tau - t}]\, \frac{\partial Y_1}{\partial\, \mathrm{Fo}_t}\, d\,\mathrm{Fo}_t\, d\xi \; (\tau > t), \tag{6-30}$$

where

$$\xi = \frac{\lambda_b L}{c_\ell \gamma_\ell g_\ell}; \quad \mathrm{Fo}_\eta = \frac{a\eta}{R_1^2} \quad (\eta = \tau,\ t,\ \tau - t);$$

$$R[\mathrm{Fo}_{\tau-t}] = -4 \sum_{n=1}^{\infty} A_n e^{-\mu_n^2 \frac{a(\tau-t)}{R_1^2}};$$

$$A_n = \frac{\pi J_1^2(k\mu_n)}{J_1^2(k\mu_n) - J_0^2(\mu_n)}; \quad k = \frac{R_2}{R_1}; \tag{6-31}$$

g_ℓ is the volumetric flow rate of the water; λ_b is the heat conductivity of the concrete; c_ℓ and γ_ℓ are the specific heat capacity and density of the water.

The integral equation (6-30) is solved by the method of iterations. As the first approximation, we can assume the value of variable $Y_{1,0}$ where $\tau = 0$. It is

$$Y_{1,0}(0, L) = 1 - e^{-R_0 \xi},$$

where

$$R_0 = -4 \sum_{n=1}^{\infty} A_n.$$

In order to determine $Z_1(\tau, L)$ and $X_1(\tau, L)$, we have

$$Z_1 = Y_1 + \int_0^{\mathrm{Fo}_\tau} M[\mathrm{Fo}_{\tau-t}] \frac{\partial Y_1}{\partial \mathrm{Fo}_t} d\,\mathrm{Fo}_t, \tag{6-31'}$$

where

$$M[\mathrm{Fo}_{\tau-t}] = -\frac{4}{\pi(k^2-1)} \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2} e^{-\mu_n^2 \frac{a(\tau-t)}{R_1^2}} \tag{6-32}$$

and

$$X_1 = \frac{1}{\xi} \int_0^{\xi} Z_1 d\xi. \tag{6-33}$$

Consequently, the water temperature at the outlet

$$T_{\text{в}L} = T_{\text{в}0} + (T_{\text{б}0} - T_{\text{в}0})\,Y_1;$$

the mean temperature of the concrete cylinder of length L

$$\bar{T}_{\text{б}L} = T_{\text{в}0} + (T_{\text{б}0} - T_{\text{в}0})\,X_1;$$

the mean temperature of the concrete in cross section L

$$T_{\text{б}L} = T_{\text{в}0} + (T_{\text{б}0} - T_{\text{в}0})\,Z_1.$$

2. Temperature calculations in the first stage of pipe cooling. Let us assume that the intensity of heat liberation in the concrete depends exponentially on time

$$q = q_0 e^{-m\tau}.$$

We note that in this case the maximum adiabatic rise in the temperature of the concrete is

$$T_{\text{ад}\,m} = \frac{q_0}{mc\gamma}.$$

a) The initial temperature of the concrete and the temperature of the water at the inlet are equal to 0. We introduce variables X_2, Y_2, Z_2, equal to

$$X_2 = \frac{\bar{T}_{\text{б}L}}{T_{\text{ад}\,m}};\quad Y_2 = \frac{T_{\text{в}L}}{T_{\text{ад}\,m}};\quad Z_2 = \frac{T_{\text{б}L}}{T_{\text{ад}\,m}}.$$

These variables are defined by the following equations

$$Y_2 = S - \int_0^{\xi}\int_0^{Fo_2} R[Fo_2]\,\frac{dY_2}{d\,Fo_2}\,d\,Fo_2\,d\xi; \qquad (6\text{-}34)$$

$$Z_2 = \bar{V} + Y_2 - \int_0^{Fo_t} M[Fo_t] \frac{\partial Y_2}{\partial Fo_t} dFo_t; \qquad (6\text{-}35)$$

$$X_2 = \frac{1}{\xi} \int_0^{\xi} Z_2 d\xi. \qquad (6\text{-}36)$$

Here

$$S = 2\pi m^* w_1 e^{-m\tau} + \sum_{n=1}^{\infty} \frac{m^{*2} A_n}{m^{*2} - \mu_n^2} e^{-\mu_n^2 \frac{a\tau}{R_1^2}};$$

$$\bar{V} = \frac{1}{(k^2 - 1)} \left[\bar{v} e^{-m\tau} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{m^{*2} A_n}{\mu_n^2 (m^{*2} - \mu_n^2)} e^{-\mu_n^2 \frac{a\tau}{R_1^2}} \right];$$

$$\bar{v} = \frac{Y_1(m^*) J_1(km^*) - J_1(m^*) Y_1(km^*)}{m^* [J_0(m^*) Y_1(km^*) - Y_0(m^*) J_1(km^*)]} - 1;$$

$$m^{*2} = \frac{mR_1^2}{a}, \quad k = \frac{R_2}{R_1}.$$

The functions R[Fo_t], M[Fo_t] are given by formulas (6-31) and (6-32), while the function w_1 is determined by formula (6-14).

From this

$$T_{\ell L} = T_{ad\,m} Y_2, \quad T_{bL} = T_{ad\,m} Z_2, \quad T_{bL} = T_{ad\,m} X_2.$$

b) General case. The initial temperature of the concrete is T_0; the temperature of the water at the inlet is $T_{\ell 0}$; heat liberation occurs in the concrete, the intensity of which depends exponentially on time

$$q = q_0 e^{-m\tau}.$$

The water temperature in cross section L

$$T_{\ell L} = T_{\ell 0} + (T_{b0} - T_{\ell 0}) Y_1 + T_{ad\,m} Y_2;$$

The mean temperature of the concrete in section L

$$T_{\text{б}L} = T_{\text{в}0} + (T_{\text{б}0} - T_{\text{в}0}) Z_1 + T_{\text{ад}\,m} Z_2;$$

The mean temperature of the concrete cylinder of length L

$$\bar{T}_{\text{б}L} = T_{\text{в}0} + (T_{\text{б}0} - T_{\text{в}0}) X_1 + T_{\text{ад}\,m} X_2.$$

In [14, 172, 173], graphs are presented for the functions X_j, Y_j and Z_j ($j = 1, 2$) for values of $k = R_2/R_1 = 100$. For values of k between 10 and 100, in determining X_j, Y_j and Z_j, we should introduce a frictionless coefficient of temperature conductivity of the concrete a_ϕ, defined by the formula

$$a_\phi = a \frac{\ln 100}{\ln k}, \quad 10 \leqslant k \leqslant 100.$$

6-4. Model of Linear Heat Sources (Sinks)

One serious difficulty encountered in the method described in the previous paragraph, based on the model of an unlimited hollow cylinder, is consideration of heat exchange with the environment through the outer surfaces of the concrete masses. One means of resolving this difficulty is to replace the pipes with linear heat sources (sinks) and solve the corresponding heat conductivity problem analytically for the area studied.

The power of the heat sources is established by calculation using the hollow cylinder method. It can also be assigned on the basis of other considerations.

We analyze below the analytic solution of certain problems of heat conductivity for masses with individual linear heat sources[1] with various types of dependence of source power on time and coordinates.

The edge conditions of the problem are homogeneous (0 initial and boundary conditions).

The sums and combinations of solutions of this paragraph, constructed in the corresponding manner, in accordance with the solutions produced in Chapter 5, cover most practically important cases, which may be encountered in designing specific water engineering projects.

[1] A sink is a negative source, differing only in the sign in the expression for power.

Spatial Temperature Field of Mass with Horizontal Linear Heat Sources

A diagram of a mass with a heat source is shown in Figure 6-2.

The differential equation

$$\frac{\partial T}{\partial \tau}=a\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{1}{c\gamma}q(y,\tau)\,\delta(z-z_0)\,\delta(x-x_0)\,\varepsilon(y)$$

$$(0<z<\infty,\ 0<x<L,\ 0<y<D,\ \tau>0), \tag{6-37}$$

where δ is the delta-function;
$\varepsilon(y)$ is a function equal to

$$\varepsilon(y)=\begin{cases}1 & \text{where } y_1<y<y_2,\\ 0 & \text{where } -y_1<y<y_1,\ -D<y<-y_2,\ y_2<y<D;\end{cases}$$

(z_0, x_0, y_1) and (z_0, x_0, y_2) are the coordinates of the end of the source.

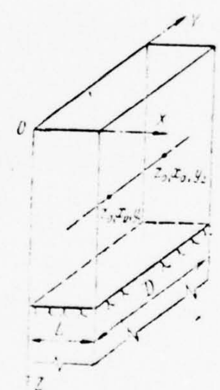


Figure 6-2. Diagram of Mass with Horizontal Linear Heat Source

Due to the obvious symmetry of the temperature field on the coordinate surfaces ZOY and ZOX homogeneous boundary conditions of the second kind are assigned, while homogeneous boundary conditions of the third kind are assigned for the remaining faces of the mass.

A nonzero value of the coordinate of the end of the source y_1 is used only for purposes of generality. In problems concerning pipe cooling of a mass and the corresponding solution, we should use $y_1 = 0$.

The solution

$$T = \frac{1}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{\|U_0\|^2 \|V_0\|^2} \cos \mu_p \frac{x_0}{L} \times$$
$$\times \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} \times$$
$$\times \int_0^{\tau} \int_{y_1}^{y_2} q(y, t) W(z, z_0, \tau - t) \exp[-aK_{pr}(\tau - t)] \cos \varkappa_r \frac{y}{D} \, dt \, dy, \tag{6-38}$$

where

$$W(z, z_0, \tau - t) = \frac{1}{2\sqrt{\pi a(\tau - t)}} \left(\exp\left[-\frac{(z - z_0)^2}{4a(\tau - t)} \right] + \right.$$
$$\left. + \exp\left[-\frac{(z + z_0)^2}{4a(\tau - t)} \right] \right) - h_z \exp[h_z^2 a(\tau - t) +$$
$$+ h_z(z + z_0)] \operatorname{erfc}\left[\frac{z + z_0}{2\sqrt{a(\tau - t)}} + h_z \sqrt{a(\tau - t)} \right];$$

$\|U_0\|^2$ and $\|V_0\|^2$ are the squares of the norms of the corresonding functions, equal to

$$\|U_0\|^2 = \frac{L}{2} \frac{\mathrm{Bi}_x^2 + \mathrm{Bi}_x + \mu_p^2}{\mathrm{Bi}_x^2 + \mu_p^2}; \quad \mathrm{Bi}_x = h_x L;$$
$$\|V_0\|^2 = \frac{D}{2} \frac{\mathrm{Bi}_y^2 + \mathrm{Bi}_y + \varkappa_r^2}{\mathrm{Bi}_y^2 + \varkappa_r^2}; \quad \mathrm{Bi}_y = h_y D; \quad K_{pr} = \frac{\mu_p^2}{L^2} + \frac{\varkappa_r^2}{D^2}.$$

Let us study certain particular cases of dependence of source power $q(y, \tau)$ on time τ and coordinate y.

1) $q(y, \tau) = q_0 = \mathrm{const.}$

$$T = \frac{q_0}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{2D}{\varkappa_r \|U_0\|^2 \|V_0\|^2} \cos \mu_p \frac{x_0}{L} \sin \varkappa_r \frac{y_2 - y_1}{D} \sin \varkappa_r \frac{y_2 + y_1}{D} \times$$
$$\times \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} L_{pr}(z, z_0, \tau), \tag{6-39}$$

where
$$L_{pr}(z, z_0, \tau) = -\frac{h_z}{a(h_z^2 - K_{pr})} \exp[(h_z^2 - K_{pr}) a\tau + h_z(z + z_0)] \times$$
$$\times \operatorname{erfc}\left[\frac{z - z_0}{2\sqrt{a\tau}} + h_z\sqrt{a\tau}\right] + \frac{1}{4a\sqrt{K_{pr}}} \sum_{i=1}^{2} (-1)^{i-1} \times$$
$$\times \Big\{ \exp[(-1)^i (z - z_0)\sqrt{K_{pr}}] \operatorname{erfc}\left[\frac{z - z_0}{2\sqrt{a\tau}} + (-1)^i \sqrt{aK_{pr}\tau}\right] +$$
$$+ \frac{1}{(h_z^2 - K_{pr})} (2h_z\sqrt{K_{pr}} + (-1)^i (h_z^2 - K_{pr})) \exp[(-1)^i (z + z_0)\sqrt{K_{pr}}] \times$$
$$\times \operatorname{erfc}\left[\frac{z + z_0}{2\sqrt{a\tau}} + (-1)^i \sqrt{aK_{pr}\tau}\right].$$

2) $q(y, \tau) = q_0 e^{-\alpha\tau}$.
$$T = \frac{q_0}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{2D}{\varkappa_r \|U_0\|^2 \|V_0\|^2} \cos\mu_p \frac{x_0}{L} \sin\varkappa_r \frac{y_2 - y_1}{D} \sin\varkappa_r \frac{y_2 + y_1}{D} \times$$
$$\times \cos\mu_p \frac{x}{L} \cos\varkappa_r \frac{y}{D} \exp[-aK_{pr}\tau] J_{pr}(z, \tau),$$

(6-40)

where

$$J_{pr}(z, \tau) = -\frac{h_z}{h_z^2 a + \alpha - K_{pr}} \exp[h_z^2 a\tau + h_z(z + z_0)] \times$$
$$\times \operatorname{erfc}\left[\frac{z + z_0}{2\sqrt{a\tau}} + h_z\sqrt{a\tau}\right] - \frac{1}{2\sqrt{\pi a}} I_{1/2}\left(\alpha - aK_{pr}, -\left(\frac{z - z_0}{2\sqrt{a}}\right)^2, \tau\right) -$$
$$- \frac{h_z^2 a - \alpha + aK_{pr}}{2\sqrt{\pi a}(h_z^2 a + \alpha - aK_{pr})} I_{1/2}\left(\alpha - aK_{pr}, -\left(\frac{z + z_0}{2\sqrt{a}}\right)^2, \tau\right) +$$
$$+ \frac{h_z}{\sqrt{\pi}(h_z^2 a + \alpha - aK_{pr})} I_{3/2}\left(\alpha - aK_{pr}, -\left(\frac{z + z_0}{2\sqrt{a}}\right)^2, \tau\right);$$

the algorithms for calculation of the function $I_{(2k+1)/2}(\alpha - aK_{pr}, -(\frac{N}{2\sqrt{a}})^2, \tau)$ (k = 1, 2) are given in § 5-2.

3) $q(y, \tau) = q_0 e^{-\alpha\tau} e^{-\beta y}$.
$$T = \frac{q_0}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{C_r}{\|U_0\|^2 \|V_0\|^2} \cos\mu_p \frac{x_0}{L} \cos\mu_p \frac{x}{L} \cos\varkappa_r \frac{y}{D} \times$$
$$\times \exp[-aK_{pr}\tau] J_{pr}(z, \tau), \quad (6\text{-}41)$$

where

$$C_r = \sum_{i=1}^{2} (-1)^i \frac{e^{-\beta y_i}}{\beta^2 + \frac{\varkappa_r^2}{D^2}} \left(\frac{\varkappa_r}{D} \sin \varkappa_r \frac{y_i}{D} - \beta \cos \varkappa_r \frac{y_i}{D} \right).$$

4) $q(y, \tau) = q_0 e^{-\alpha\tau} y e^{-\beta y}$.

The solution is the same as in (6-41), but

$$C_r = \sum_{i=1}^{2} (-1)^i \frac{e^{-\beta y_i}}{\beta^2 + \frac{\varkappa_r^2}{D^2}} \left[\frac{\varkappa_r}{D} y_i + \frac{2\frac{\varkappa_r \beta}{D}}{\beta^2 + \frac{\varkappa_r^2}{D^2}} \sin \varkappa_r \frac{y_i}{D} - \right.$$

$$\left. - \left(\beta y_i - \frac{\beta^2 - \frac{\varkappa_r^2}{D^2}}{\beta^2 + \frac{\varkappa_r^2}{D^2}} \right) \cos \varkappa_r \frac{y_i}{D} \right].$$

Spatial Temperature Fields of a Mass with Vertical Linear Heat Sources (Figure 6-3)

The differential equation

$$\frac{\partial T}{\partial \tau} = a \left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{1}{c\gamma} q(z, \tau) \delta(x - x_0) \delta(y - y_0) \varepsilon(z), \qquad (6\text{-}42)$$

where

$$\varepsilon(z) = \begin{cases} 1 & \text{where } z_1 < z < z_2, \\ 0 & \text{where } 0 < z < z_1 \text{ и } z_2 < z < \infty. \end{cases}$$

The solution

$$T = \frac{1}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{\|U_0\|^2 \|V_0\|^2} \cos \mu_p \frac{x_0}{L} \times$$

$$\times \cos \varkappa_r \frac{y_0}{D} \cos \mu_p \frac{x}{L} \times$$

$$\times \cos \varkappa_r \frac{y}{D} \int_0^{\tau} \int_{z_1}^{z_2} q(\eta, t) W(z, \eta, \tau - t) \times$$

$$\times \exp[-aK_{pr}(\tau - t)]\, dt\, d\eta. \qquad (6\text{-}43)$$

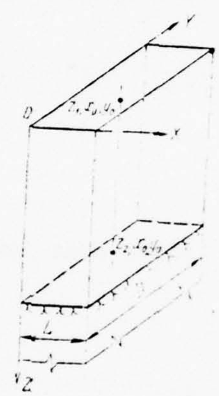


Figure 6-3. Diagram of Mass with Vertical Linear Heat Source

Suppose $q = q_0$ = const. Then

$$T = \frac{q_0}{c\gamma} \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{\|U_0\|^2 \|V_0\|^2} \cos \mu_p \frac{x_0}{L} \cos \varkappa_r \frac{y_0}{D} \times$$

$$\times \cos \mu_p \frac{x}{L} \cos \varkappa_r \frac{y}{D} S_{pr}(z, \tau),$$

where

$$S_{pr}(z,\tau)=\sum_{i=1}^{2}(-1)^{i+1}\Big\{\exp[-aK_{pr}\tau]\Big(\frac{1}{2aK_{p}}\operatorname{erfc}\Big[\frac{z+z_i}{2\sqrt{a\tau}}\Big]+$$

$$+\frac{1}{2aK_{pr}}\operatorname{erfc}\Big[\frac{z-z_i}{2\sqrt{a\tau}}\Big]+\frac{1}{h_z^2a-aK_{pr}}\exp[h_z^2a\tau+h_z(z+z_i)]\times$$

$$\times\operatorname{erfc}\Big[\frac{z+z_i}{2\sqrt{a\tau}}+h_z\sqrt{a\tau}\Big]\Big)-\frac{1}{4aK_{pr}}\Big(\exp[-(z-z_i)\sqrt{K_{pr}}]\times$$

$$\times\operatorname{erfc}\Big[\frac{z-z_i}{2\sqrt{a\tau}}-\sqrt{aK_{pr}\tau}\Big]+\exp[(z-z_i)\sqrt{K_{pr}}]\operatorname{erfc}\Big[\frac{z-z_i}{2\sqrt{a\tau}}+$$

$$+\sqrt{aK_{pr}\tau}\Big]+\frac{h_z-\sqrt{K_{pr}}}{h_z+\sqrt{K_{pr}}}\exp[-(z+z_i)\sqrt{K_{pr}}]\operatorname{erfc}\Big[\frac{z+z_i}{2\sqrt{a\tau}}-$$

$$-\sqrt{aK_{pr}\tau}\Big]+\frac{h_z+\sqrt{K_{pr}}}{h_z-\sqrt{K_{pr}}}\exp[(z+z_i)\sqrt{K_{pr}}]\times$$

$$\times\operatorname{erfc}\Big[\frac{z+z_i}{2\sqrt{a\tau}}+\sqrt{aK_{pr}\tau}\Big]\Big)\Big\}. \tag{6-44}$$

Planar Temperature Field with Linear Heat Source

The differential equation

$$\frac{\partial T}{\partial\tau}=a\left(\frac{\partial^2T}{\partial z^2}+\frac{\partial^2T}{\partial x^2}\right)-HT+\frac{1}{c\gamma}q(\tau)\,\delta(z-z_0)\,\delta(x-x_0)$$
$$(0<z<\infty,\ 0<x<L,\ \tau>0). \tag{6-45}$$

The solution

$$T=\frac{1}{c\gamma}\sum_{p=1}^{\infty}\frac{\cos\mu_p\frac{x_0}{L}}{\|U_0\|^2}\cos\mu_p\frac{x}{L}\int_0^{\tau}q(t)\exp[-aK_{pH}(\tau-t)]\,W(z,z_0,\tau-t)\,dt. \tag{6-46}$$

Particular cases

1) $q=q_0=\text{const}.$

$$T=\frac{q_0}{c\gamma}\sum_{p=1}^{\infty}\frac{1}{\|U_0\|^2}\cos\mu_p\frac{x_0}{L}L_{pH}(z,\tau)\cos\mu_p\frac{x}{L}. \tag{6-47}$$

2) $q = q_0 e^{-\alpha\tau}$.

$$T = \frac{q_0}{c\gamma} \sum_{p=1}^{\infty} \frac{1}{\|U_0\|^2} \cos \mu_p \frac{x_0}{L} \cos \mu_p \frac{x}{L} J_{pH}(z, \tau). \qquad (6\text{-}48)$$

Here $L_{pH}(z, \tau)$ and $J_{pH}(z, \tau)$ differ from $L_{pr}(z, \tau)$ and $J_{pr}(z, \tau)$, presented earlier, in that here K_{pr} can be replaced by $K_{pH} = (\mu_p^2/L^2) + (H/a)$.

6-5. Model of Linear Temperature Sources

The model of linear temperature sources is based on assignment of temperature on a line corresponding to the axis of the coil. This modeling is correct, since the diameter of the pipe is significantly less than the spacing between pipes.

A study is made of a two-dimensional temperature field in a plane perpendicular to the axes of the pipe.

With this approach, in addition to the usual statement of the problem, considering heat liberation in the concrete and its dependence on temperature and time, the initial and boundary conditions, etc., at points of intersection of the axes of the pipe with the plane in which calculation is performed (calculation plane), the cooling water temperature is assigned.

The solution of the problem of the temperature mode of the mass is performed by the method of finite differences.

When this is done, we can utilize the method and correspondingly the program of calculations of a two-dimensional temperature field of a discretely growing concrete mass by computer as presented in Chapter 5. The program in question requires only slight changes: at each time step for the period of action of the cooling system, the water temperature is entered into the computer memory locations corresponding to points of intersection of pipe axes with the calculation plane.

In order to calculate the temperature at the boundary points and on the lines where the blocks meet, we use algorithms following from the thermal balance equation for elementary sectors near the corresponding boundaries of the body. The advantage of these grid functions is the fact that the "thermal" interaction of the contour points is considered both along the boundary and with neighboring internal points. This last feature is quite important, particularly when there are points with assigned temperature along the contour of the body. In the case which we are studying, this occurs when cooling pipes are laid in horizontal structural seams, at the interfaces between blocks, etc.

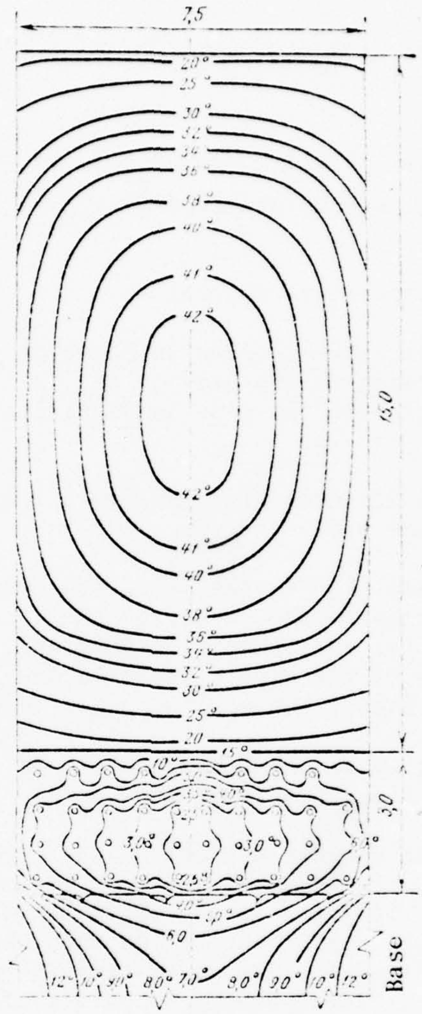


Figure 6-4. Temperature Field of a Mass with Pipe Cooling (Row Placement of Pipe)

Figures 6-4 and 6-5 present certain results of calculation of planar temperature fields in concrete masses with pipe cooling.

A rectangular grid is placed on the calculation area in such a way that the points of intersection of pipe axes with the calculation plane fall at its junctions. As a rule, there are 5 to 6 grid junctions between these points.

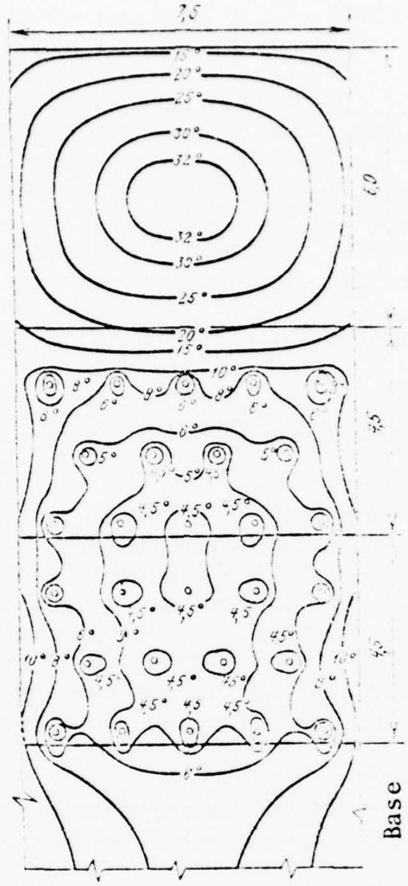


Figure 6-5. Temperature Field of a Mass of Pipe Cooling (Checkerboard Placement of Pipes)

The isotherms of Figure 6-4 are constructed for a mass consisting of two blocks: a three-meter block at the bottom with pipe cooling and a naturally cooled block 15 m high located above it. The width of the mass is 7.5 m. The blocks were poured simultaneously. The placement of the cooling pipes is of row type with a vertical and horizontal spacing of 0.75 m. Water with an initial temperature of 2 C enters the cooling system near the axis of the mass and flows out at its periphery. The heating of the cooling fluid is considered, in that it is assumed that its temperature increases as it passes through the coil by 4 C during the first 15 days, by 3 C during the next 15 days, then by 2 C.

Figure 6-5 shows the distribution of temperature in the concrete mass with checkerboard placement of cooling pipe. The mass consists of 3 blocks. The bottom two blocks, each 4.5 m high, are cooled with a system of pipes

with a spacing of 1.5 m, while the height of the top, naturally cooled block, is 6 m. The temperature of the cooling water is assumed equal to 2 C.

In calculation of the temperature fields of the concrete masses with pipe cooling by the method of linear temperature sources, achievement of sufficient calculation accuracy requires a very small spacing in the coordinates. This results from the relatively small spacing between pipes (0.75-1.5 m). Due to the small steps along the coordinates, calculation of one block requires a large number of points in the difference grid and, consequently, many computer memory locations. Thus, for an artificially cooled block 3 m high and 15 m wide with a spacing between pipes of 1.5 m, the difference grid should contain some 900 junctions (with but 6 junctions between pipes).

Still greater difficulties arise if we use an explicit finite difference plan, since the small spacing with respect to the coordinates leads to small time steps.

In many cases, however, we can construct calculation plans which allow us to reduce the volume of calculations significantly. For example, let us study the thermal state of an artificially cooled concrete column of sufficient width (Figure 6-6). The temperature of the middle portion of the column, remote from the side surfaces, is determined basically by the effect of the cooling pipes and heat exchange with the horizontal surfaces. Cooling from the lateral surfaces influences the temperature mode only in those areas located near the boundaries and encompasses several vertical rows of pipes.

The vertical axes of symmetry in the middle portion of the column can be used to separate a calculation area shown in Figure 6-6a.

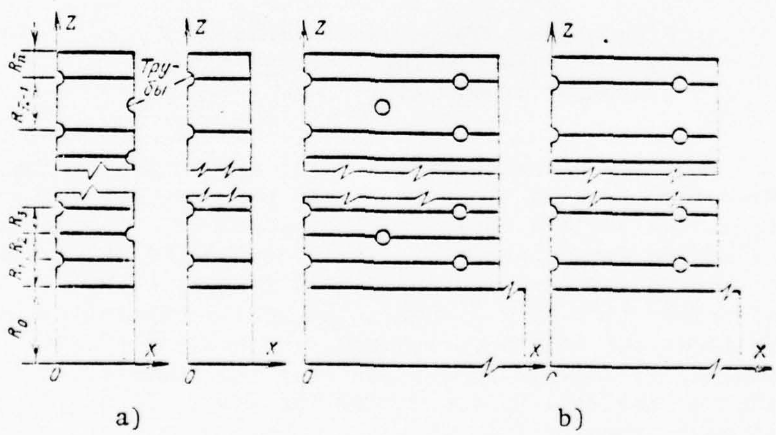


Figure 6-6. Plan of Column with Pipe Cooling (Simplified Calculation Version). a, Diagram of Calculation Areas in Middle Portion of Mass; b, Diagram of Calculation Areas Near Lateral Boundaries of Mass

The width of the calculation area is equal to half the spacing between pipes in the horizontal direction. On the vertical boundaries of the area, symmetry conditions are assigned -- homogeneous boundary conditions of the second kind.

The calculation plans in Figure 6-6b are used to determine the temperature near the side boundaries of the mass. The width of the calculation area is selected as a multiple of a half space between pipes. Depending on the value of the multiple, this area contains one, two or more vertical rows of pipes. At the left boundary conditions of symmetry are assigned, at the right boundary -- boundary conditions of the third kind.

In calculating the temperature field of discretely growing concrete masses based on these calculation plans, we can use the algorithms of Chapter 5 with the supplements presented in this section.

CHAPTER 7. METHODS OF CALCULATION OF TEMPERATURE FIELDS OF CONCRETE STRUCTURES DURING THE PERIOD OF USE

Let us recall the basic statements of § 2-3, in which we formulated the problem of heat conductivity for concrete structures and their elements during the period of use. During the period of use, two modes are distinguished: the transient and quasistable modes.

In the transient mode, the temperature field is established as a result of solution of the equation

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T \tag{7-1}$$

with assigned initial and boundary conditions.

In order to determine the temperature field of a structure in the quasi-stable mode, we solve equation (7-1) with assigned boundary conditions, but without initial conditions.

7-1. Calculations of Temperature Fields of Structural Elements During the Period of Operation

For simplicity of presentation of the material, we assume that the temperature of the medium changes with time according to the rule

$$\psi(\tau) = T_c + A_c \sin(\omega\tau + \varepsilon), \tag{7-2}$$

where T_c is the mean temperature of the medium during the period θ; A_c is the amplitude of temperature fluctuations; $\omega = 2\pi/\theta$ is the oscillating frequency; ε is the initial phase.

Transient Temperature Mode

1. Semilimited body ($0 < x < \infty$). The initial temperature of the body is $f(x)$. The ambient temperature $\psi(\tau) = T_c + A_c \sin(\omega\tau + \varepsilon_c)$ [or of the surface $\phi(\tau) = T_\pi + A_\pi \sin(\omega\tau + \varepsilon_\pi)$].

Let us assume that temperature function $T(x, \tau)$ is equal to

$$T(x, \tau) = T_1(x, \tau) + T_2(x, \tau), \tag{7-3}$$

where $T_1(x, \tau)$ is the solution of equation (7-1) with initial body temperature $f(x)$ and ambient (surface) temperature $T_c(T_\pi)$; $T_2(x, \tau)$ is the solution of equation (7-1) with zero initial temperature of the body and ambient (surface) temperature equal to the harmonic component $A_c = \sin(\omega\tau + \varepsilon_c)$ ($A_\pi \sin(\omega\tau + \varepsilon_\pi)$).

Then function $T_1(x, \tau)$ is determined by the expression

$$T_1(x, \tau) = \int_0^\infty [f(x_0) - T_c] \left\{ \frac{1}{2\sqrt{\pi a\tau}} \left(\exp\left[-\frac{(x - x_0)^2}{4a\tau} \right] + \exp\left[-\frac{(x + x_0)^2}{4a\tau} \right] \right) - h \exp[h^2 a\tau + h(x + x_0)] \operatorname{erfc}\left[\frac{x + x_0}{2\sqrt{a\tau}} + h\sqrt{a\tau} \right] \right\} dx_0^{1} \tag{7-4'}$$

(boundary condition of the third kind);

$$T_1(x, \tau) = \frac{1}{2\sqrt{\pi a\tau}} \int_0^\infty [f(x_0) - T_\pi] \left(\exp\left[-\frac{(x - x_0)^2}{4a\tau} \right] - \exp\left[-\frac{(x + x_0)^2}{4a\tau} \right] \right) dx_0 \tag{7-4}$$

(boundary condition of the first kind).

In the particular case where $f(x) = T_0 = \text{const}$

$$T_1(x, \tau) = T_0 + (T_c - T_0) \left(\operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}} \right] - \exp[h^2 a\tau + hx] \times \operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}} + h\sqrt{a\tau} \right] \right)$$

(boundary condition of the third kind)

$$T_1(x, \tau) = T_0 + (T_c - T_0) \operatorname{erfc}\left[\frac{x}{2\sqrt{a\tau}} \right]$$

(boundary condition of the first kind).

Function $T_2(x, \tau)$ is:

---

[1] $h = \alpha/\lambda$ is the relative heat transfer factor.

$$T_2(x, \tau) = \frac{hA_c}{\sqrt{k^2 + (k+h)^2}} e^{-kx} \sin(\omega\tau + \varepsilon_c - kx - \delta) +$$

$$+ \frac{2ahA_c}{\pi} \int_0^\infty \frac{(\omega \cos \varepsilon_c - au^2 \sin \varepsilon_c)(u \cos ux + h \sin ux)}{(a^2u^2 + \omega^2)(h^2 + u^2)} e^{-au^2\tau} u\, du \qquad (7\text{-}5')$$

(boundary condition of the third kind);

$$T_2(x, \tau) = A_{\text{п}} e^{-kx} \sin(\omega\tau + \varepsilon_{\text{п}} - kx) +$$

$$+ \frac{2aA_{\text{п}}}{\pi} \int_0^\infty \frac{\omega \cos \varepsilon_{\text{п}} - au^2 \sin \varepsilon_{\text{п}}}{\omega^2 + a^2u^4} \sin ux e^{-au^2\tau} u\, du \qquad (7\text{-}5)$$

(boundary condition of the first kind).

Here

$$k = \sqrt{\frac{\omega}{2a}}; \quad \delta = \operatorname{arctg} \frac{k}{k+h}.$$

Note. Solutions (7-5) and (7-5') were produced by the method of the Laplace transform [54].

2. Wall $(0 < x < R)$, hollow cylinder $(0 < r < R)$, hollow cylinder $(R_1 < r < R_2)$. The initial temperature of the body $f(\xi)$. At the surface of the body for definition we assign boundary conditions of the third kind

$$\frac{\partial T(R_j, \tau)}{\partial \xi} = (-1)^j h_j [\varphi_j(\tau) - T(R_j, \tau)] \quad (j = 1, 2),$$

where

$$\varphi_j(\tau) = T_{jc} + A_{jc} \sin(\omega_j\tau + \varepsilon_j) = T_{jc} + \operatorname{Im} A_{jc} e^{i(\omega_j\tau + \varepsilon_j)}.$$

Subsequently, we will use a complex form, represented by the harmonic component of the ambient temperature, omitting the symbol for the imaginary portion Im. Obviously, the harmonic component of the temperature function with this representation will be equal to the imaginary portion Im of the corresponding solution.

The solution to the problem of the temperature field of the body in the transient mode can be produced using a method of finite integral transforms by means of substitution function (see § 3.3).

Let us assume:

$$T(\xi,\tau)=\Phi_{I}(\xi,\tau)+\Phi_{II}(\xi,\tau)-\theta(\xi,\tau),$$

where Φ_I (ξ, τ) and $\Phi_{II}(\xi, \tau)$ are substitution functions with F functions of the first and second kinds.

The substitution function is selected as

$$\Phi_{I}(\xi,\tau)=T_{2c}F_{Ia}(\xi)+T_{1c}F_{Ib}(\xi);$$
$$\Phi_{II}(\xi,\tau)=A_{2c}e^{i(\omega_2\tau+\varepsilon_2)}F_{IIa}(\xi)+A_{1c}e^{i(\omega_1\tau+\varepsilon_1)}F_{IIa}(\xi).$$

Here the F functions are solutions of the differential equations:

F function of first kind

$$\frac{1}{\xi^{s}}\frac{d}{d\xi}\left(\xi^{s}\frac{dF_{I}}{d\xi}\right)=0\ (R_1<\xi<R_2,\ s=0\vee 1); \tag{7-6'}$$

F function of second kind

$$\frac{1}{\xi^{s}}\frac{d}{d\xi}\left(\xi^{s}\frac{dF_{II}}{d\xi}\right)-i\frac{\omega}{a}F_{II}=0\ (R_1<\xi<R_2,\ s=0\vee 1). \tag{7-6}$$

satisfying the specific boundary conditions:

Functions F_{Ia} and F_{IIa}

$$\frac{dF_{a}(R_1)}{d\xi}=h_1F_{a}(R_1);\quad \frac{dF_{a}(R_2)}{d\xi}=h_2[1-F_{a}(R_2)]; \tag{7-7'}$$

functions F_{Ib} and F_{IIb}

$$\frac{dF_{b}(R_1)}{d\xi}=-h_1[1-F_{b}(R_1)];\ \frac{dF_{b}(R_2)}{d\xi}=-h_2F_{b}(R_2). \tag{7-7}$$

Then to define function $\theta(\xi, \tau)$ we have

$$\frac{\partial\vartheta}{\partial\tau} = a\frac{1}{\xi^s}\frac{\partial}{\partial\xi}\left(\xi^s\frac{\partial\vartheta}{\partial\xi}\right) \quad (R_1 < \xi < R_2,\ s = 0 \vee 1,\ \tau > 0);$$
$$\vartheta(\xi, 0) = T_{2c}F_{1a}(\xi) + T_{1c}F_{1b}(\xi) + A_{2c}e^{i\varepsilon_2}F_{11a}(\xi) +$$
$$+ A_{1c}e^{i\varepsilon_1}F_{11b}(\xi) - f(\xi) \quad (R_1 \leqslant \xi \leqslant R_2);$$
$$\frac{\partial\vartheta(R_1, \tau)}{\partial\xi} = h_1\vartheta(R_1, \tau); \quad \frac{\partial\vartheta(R_2, \tau)}{\partial\xi} = -h_2\vartheta(R_2, \tau). \tag{7-8}$$

The finite integral transform of this problem yields

$$\frac{d\bar{\vartheta}_n}{d\tau} + \frac{\mu_n^2 a}{R^2}\bar{\vartheta}_n = 0; \tag{7-9}$$
$$\bar{\vartheta}_n(0) = T_{2c}\bar{F}_{1a} + T_{1c}\bar{F}_{1b} + A_{2c}e^{i\varepsilon_2}\bar{F}_{11a} + A_{1c}e^{i\varepsilon_1}\bar{F}_{11b} - \bar{f}_n. \tag{7-10}$$

The mappings of the $\overline{F}$ functions can be produced by applying finite integral transforms to equations (7-6) and (7-6') considering the boundary conditions (7-7) and (7-7').

The solution of the first order ordinary differential equation (7-9) with the condition (7-10) is

$$\bar{\vartheta}_n = (T_{2c}\bar{F}_{1a} + T_{1c}\bar{F}_{1b} + A_{2c}e^{i\varepsilon_2}\bar{F}_{11a} + A_{1c}e^{i\varepsilon_1}\bar{F}_{11b} - \bar{f}_n)\,e^{-\mu_n^2\frac{a\tau}{R^2}}.$$

From this, based on the corresponding inversion formula

$$\vartheta = \sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}(T_{2c}\bar{F}_{1a} + T_{1c}\bar{F}_{1b} + A_{2c}e^{i\varepsilon_2}\bar{F}_{11a} + A_{1c}e^{i\varepsilon_1}\bar{F}_{11b} -$$
$$- \bar{f}_n)\,U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R^2}}.$$

Here, as before, $U_0(\mu_n\frac{\xi}{R})$ is the Eigenfunction of the problem; $||U_0||^2$ is the square of the norm of the Eigenfunction; μ_n is the root of the characteristic equation; R is the characteristic diameter of the body.

Thus, the desired temperature function T(ξ, τ) is:

$$T(\xi, \tau) = T_{2c}F_{1a}(\xi) + T_{1c}F_{1b}(\xi) + \operatorname{Im} A_{2c}e^{i(\omega_2\tau+\varepsilon_2)}F_{11a}(\xi) +$$
$$+ \operatorname{Im} A_{1c}e^{i(\omega_1\tau+\varepsilon_1)}F_{11b}(\xi) - \operatorname{Im}\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}(T_{2c}\bar{F}_{1a} + T_{1c}\bar{F}_{1b} +$$
$$+ A_{2c}e^{i\varepsilon_2}\bar{F}_{11a} + A_{1c}e^{i\varepsilon_1}\bar{F}_{11b} - \bar{f}_n)\, U_0\left(\mu_n\frac{\xi}{R}\right)e^{-\mu_n^2\frac{a\tau}{R_1^2}}. \tag{7-11}$$

Quasistable Temperature Mode

1. Semilimited body ($0 < x < \infty$). Let us study only the harmonic component of temperature function T(x, τ) in expression (7-3), namely $T_2(x, \tau)$. The subscript 2 will be omitted.

With sufficiently long times, the integral terms in the right portions of expressions(7-5) and (7-5') can be ignored. The temperature field of the semilimited body in this case takes on the nature of a quasistable field

$$T(x, \tau) = \frac{hA_c}{\sqrt{k^2 + (k+h)^2}}e^{-kx}\sin(\omega\tau + \varepsilon_c - kx - \delta) \tag{7-12}$$

(boundary condition of the third kind)

$$T(x, \tau) = A_n e^{-kx}\sin(\omega\tau + \varepsilon_n - kx) \tag{7-12'}$$

(boundary condition of the first kind).

This result can be produced in a different way if we represent the oscillating process in complex form and assume:

$$T(x, \tau) = A(x)e^{i(\omega\tau+\varepsilon)}.$$

Then we produce the following differential equation for determination of A(x):

$$\frac{d^2A}{dx^2} - i\frac{\omega}{a}A = 0, \tag{7-13}$$

which should be integrated with the boundary conditions

$$\frac{dA(0)}{dx} = -h[A_0 - A(0)];\ A(\infty) = 0$$

(boundary condition of third kind),

$$A(0) = A_n;\ A(\infty) = 0$$

(boundary condition of first kind).

The temperature functions (7-12) and (7-12') are temperature waves with wavelength λ and with wave number $k = \sqrt{\omega/2a}$, where

$$\lambda = \frac{2\pi}{k} = 2\pi\sqrt{\frac{2a}{\omega}} = 2\sqrt{\pi a\theta}\ \left(\theta = \frac{2\pi}{\omega}\right. - \text{is the period}).$$

For concrete and rock bases, the temperature conductivity factor of which falls within the limits a = 0.003-0.005 m^2/hr, wavelength λ is 18-23 m for oscillations with a period θ = 1 year = 8760 hr and 0.9-1.2 m for oscillations with a period θ = 1 day = 24 hr.

We can see from formulas (7-12) and (7-12') that the amplitude of temperature fluctuations decreases with depth according to the formula

$$e^{-kx} = e^{-x\sqrt{\frac{\omega}{2a}}} = e^{-2\pi\frac{x}{\lambda}}.$$

This attenuation is quite intensive: at a distance of 1 wave $x = \lambda$, the amplitude of the head temperature oscillations decreases to $e^{-2\pi}$ = 0.0019 times its initial value.

Furthermore, there is a delay in the phase of the temperature wave of

$$kx + \delta = x\sqrt{\frac{\omega}{2a}} + \operatorname{arctan}\frac{k}{k+h}$$

with boundary conditions of the third kind and

$$kx = x\sqrt{\frac{\omega}{2a}}$$

with boundary conditions of the first kind.

Thus, the annual temperature fluctuations penetrate into the concrete mass and rock base to a depth of 10-15 m, the diurnal fluctuations -- to a depth of 0.5-0.8 m.

The temperature of the surface of the semilimited body in the quasistable mode with boundary conditions of the third kind performs harmonic oscillations according to the expression

$$T(0, \tau) = B \sin(\omega\tau + \varepsilon - \delta), \tag{7-14}$$

where

$$B = \frac{hA_0}{\sqrt{k^2 + (k+h)^2}}. \tag{7-15}$$

The ratio of oscillating amplitudes of the temperature of the surface of the body and medium is:

$$m = \frac{B}{A_0} = \frac{h}{\sqrt{k^2 + (k+h)^2}}. \tag{7-16}$$

Where α = 20 kcal/(m^2·hr·C) (normalized value of heat transfer factor from open concrete surfaces) and λ = 2.0 kcal/(m·hr·C) (mean value of heat conductivity factor of concrete), this ratio is equal to m = 0.97 for the annual temperature fluctuations of the air and m = 0.58 for daily fluctuations.

This last result is important in the respect that in calculating temperature fields of concrete bodies of sufficient width, caused by annual fluctuations in ambient temperature, it allows us to replace the boundary conditions of the third kind with simpler boundary conditions of the first kind.

2. A wall (0 < x < R). In the preceding section, we produced a solution to the problem of the temperature field of a body (wall, cylinder) with boundary conditions of the third kind [formula (7-11)]. With sufficiently great τ, the series in the right portion of formula (7-11) can be ignored.

Then the temperature function $T(\xi, \tau)$ is found to consist of two parts:

$$T_{cT} = T_{2c}F_{1a}(\xi) + T_{1c}F_{1b}(\xi),$$

corresponding to the stationary temperature field, and

$$T_{q\,\text{ct}} = \operatorname{Im} A_{2c} e^{i(\omega_2\tau+\varepsilon_2)} F_{IIa}(\xi) + \operatorname{Im} A_{1c} e^{i(\omega_1\tau+\varepsilon_1)} F_{IIb}(\xi), \qquad (7\text{-}17)$$

corresponding to a quasistable temperature field.

We will analyze only the quasistable component of the temperature field.

We can write the boundary conditions in the general form

$$\alpha_j \frac{\partial T}{\partial \xi}\bigg|_{\Gamma} + \beta_j T\big|_{\Gamma} = \gamma_j g_j(\tau) \quad (j = 1,\ 2). \qquad (7\text{-}18)$$

Depending on the type of boundary conditions, function $g_j(\tau)$ may be the assigned temperature of the medium, surface or heat flux. According to the condition, $g_j(\tau)$ is a simple harmonic function of time.

Let us represent it in complex form

$$g_j(\tau) = A_j \sin(\omega_j\tau + \varepsilon_j) = \operatorname{Im} A_j e^{i(\omega_j\tau+\varepsilon_j)} \quad (j = 1,\ 2).$$

Generalizing solution (7-17) for a quasistable temperature mode to cover the case of boundary conditions of the general form (7-18), we can write

$$T(\xi,\ \tau) = \operatorname{Im} A_{2c} e^{i(\omega_2\tau+\varepsilon_2)} F_{IIa}(\xi) + \operatorname{Im} A_{1c} e^{i(\omega_1\tau+\varepsilon_1)} F_{IIb}(\xi),$$

where the F function of the second kind is a solution of the differential equation

$$\frac{1}{\xi^s}\frac{d}{d\xi}\left(\xi^s \frac{dF_{II}}{d\xi}\right) - i\frac{\omega}{a} F_{II} = 0 \quad (s = 0 \vee 1)$$

with specific boundary conditions corresponding to the boundary conditions of the initial problem.

The wall is one of the most widespread calculation plans, and let us therefore discuss it in more detail.

a) Suppose the temperature of both wall surfaces ($-R < x < R$) changes according to the rule $A \sin(\omega\tau + \varepsilon)$.

Due to the obvious symmetry of the temperature field, we will study half the wall ($0 < x < R$), assigning on its surfaces:

the homogeneous boundary condition of the second kind where $x = 0$

$$\frac{\partial T(0, \tau)}{\partial x} = 0 \text{ (symmetry condition)}$$

and the heterogeneous boundary condition of the first kind where $x = R$

$$T(R, \tau) = \varphi_2(\tau) = A \sin(\omega\tau + \varepsilon) = \operatorname{Im} A e^{i(\omega\tau + \varepsilon)}.$$

The substitution function $\Phi(x, \tau)$ is selected in the form

$$\Phi(x, \tau) = \varphi_2 F_{IIa}(x),$$

where the F function of the second kind satisfies the differential equation

$$\frac{d^2 F_{IIa}}{dx^2} - i\frac{\omega}{a} F_{IIa} = 0$$

and the boundary conditions

$$\frac{dF_{IIa}(0)}{dx} = 0; \quad F_{IIa}(R) = 1.$$

From this

$$F_{IIa}(x) = \frac{\operatorname{ch} x\sqrt{i\frac{\omega}{a}}}{\operatorname{ch} R\sqrt{i\frac{\omega}{a}}}$$

or, since

$$\sqrt{i} = \pm(1 + i)\frac{1}{\sqrt{2}},$$

$$F_{11a}(x) = \frac{\operatorname{ch} k^* x(1+i)}{\operatorname{ch} k^* R(1+i)},$$

where $k^* = \sqrt{\omega/2a}$ is the wave number.

Thus, if the wall surface temperature changes according to the rule A sin $(\omega\tau + \varepsilon)$ (case "a"), its quasistable thermal state is described by the temperature function

$$T(x, \tau) = \operatorname{Im}\left[Ae^{i(\omega\tau+\varepsilon)}\frac{\operatorname{ch} k^* x(1+i)}{\operatorname{ch} k^* R(1+i)}\right] \tag{7-19}$$

or

$$T(x, \tau) = A_m \sin(\omega\tau + \varepsilon + \delta), \tag{7-20}$$

where

$$A_m = A\left|\frac{\operatorname{ch} k^* x(1+i)}{\operatorname{ch} k^* R(1+i)}\right| = A\left\{\frac{\operatorname{ch} 2k^* x + \cos 2k^* x}{\operatorname{ch} 2k^* R + \cos 2k^* R}\right\}^{1/2};$$

$$\delta = \arg\left\{\frac{\operatorname{ch} k^* x(1+i)}{\operatorname{ch} k^* R(1+i)}\right\};$$

$$k^* = \left(\frac{\omega}{2a}\right)^{1/2}.$$

Let us present a summary of the basic formulas for certain other calculation cases.

b) A wall $(0 < x < R)$; the temperature of the surface $x = 0$ is 0, the temperature of the surface $x = R$ changes according to the rule A sin $(\omega\tau + \varepsilon)$.

$$T(x, \tau) = A_m \sin(\omega\tau + \varepsilon + \delta), \tag{7-21}$$

where

$$A_m = A\left|\frac{\operatorname{sh} k^* x(1+i)}{\operatorname{sh} k^* R(1+i)}\right| = A\left\{\frac{\operatorname{ch} 2k^* x - \cos 2k^* x}{\operatorname{ch} 2k^* R - \cos 2k^* R}\right\}^{1/2};$$

$$\delta = \arg\left\{\frac{\operatorname{sh} k^* x(1+i)}{\operatorname{sh} k^* R(1+i)}\right\}.$$

c) A wall (-R < x < R); the ambient temperature (boundary condition of the third kind) where x = -R and x = R is replaced according to the rule A sin (ωτ + ε).

$$T(x, \tau) = \frac{hAM_0}{M_1} \sin(\omega\tau + \varepsilon + \delta_0 - \delta_1), \tag{7-22}$$

where

$$M_0 e^{i\delta_0} = \operatorname{ch} k^*x \cos k^*x + i \operatorname{sh} k^*x \sin k^*x;$$
$$M_1 e^{i\delta_1} = k^* \operatorname{sh} k^*R \cos k^*R - k^* \operatorname{ch} k^*R \sin k^*R + h \operatorname{ch} k^*R \cos k^*R + $$
$$+ i (k^* \operatorname{sh} k^*R \cos k^*R + k^* \operatorname{ch} k^*R \sin k^*R + h \operatorname{sh} k^*R \sin k^*R).$$

The solutions are given in these forms in [54].

In all of the formulas presented above we encounter the dimensionless quantity k*R, sometimes called the dimensionless thickness of the wall.

As the calculations show, where k*R > 5 the temperature perturbations of one surface of the wall practically do not reach its other surface. This provides a basis under these conditions for analysis of the walls as a semilimited body and for use of the corresponding calculation formulas. We also arrived at similar results in analysis of the quasistable temperature field in an unlimited body.

3. Solid cylinder (0 < r < R).

a) Boundary conditions of the first kind

$$T(R, \tau) = \operatorname{Im} Ae^{i(\omega\tau + \varepsilon)}; \quad T(0, \tau) - \text{is finite.}$$

The temperature function T(r, τ) is equal to:

$$T(r, \tau) = \operatorname{Im} \left[Ae^{i(\omega\tau + \varepsilon)} \frac{I_0\left(\omega^* \sqrt{i} \frac{r}{R}\right)}{I_0(\omega^* \sqrt{i})} \right]; \quad \omega^{*2} = \frac{\omega R^2}{a}. \tag{7-23}$$

If we introduce the Kelvin functions ber z and bei z according to the known relationship [19, 160]

$$I_0(z\sqrt{i}) = \operatorname{ber} z + i \operatorname{bei} z,$$

then after simple transforms, function (7-23) is transformed to

$$T(r,\ \tau) = A_m \sin(\omega\tau + \varepsilon + \varphi), \qquad (7\text{-}23')$$

where

$$A_m = \sqrt{\frac{\operatorname{ber}^2 m^* \frac{r}{R} + \operatorname{bei}^2 m^* \frac{r}{R}}{\operatorname{ber}^2 m^* + \operatorname{bei}^2 m^*}};$$

$$\varphi = \operatorname{arctg} \frac{\operatorname{bei} m^* \frac{r}{R} \operatorname{ber} m^* - \operatorname{ber} m^* \frac{r}{R} \operatorname{bei} m^*}{\operatorname{ber} m^* \frac{r}{R} \operatorname{ber} m^* + \operatorname{bei} m^* \frac{r}{R} \operatorname{bei} m^*}.$$

b) Boundary conditions of the third kind

$$\frac{\partial T(R,\ \tau)}{\partial r} = h\left[\operatorname{Im} Ae^{i(\omega\tau+\varepsilon)} - T(R,\ \tau)\right];\ T(0,\ \tau) \text{ — is finite.}$$

The temperature function

$$T(r,\ \tau) = \operatorname{Im}\left[Ae^{i(\omega\tau+\varepsilon)} \frac{I_0\left(m^* \sqrt{i}\frac{r}{R}\right)}{I_0(m^* \sqrt{i}) + \frac{1}{\mathrm{Bi}} m^* \sqrt{i}\, I_1(m^* \sqrt{i})}\right],\ \mathrm{Bi} = hR. \qquad (7\text{-}24)$$

4. Hollow cylinder ($R_1 < r < R_2$).

a) Boundary conditions of the first kind

$$T(R_1,\ \tau) = 0,\ T(R_2,\ \tau) = \operatorname{Im} Ae^{i(\omega\tau+\varepsilon)}.$$

The temperature function

$$T(r,\ \tau) = \operatorname{Im}\left[Ae^{i(\omega\tau+\varepsilon)} \frac{K_0(\omega^* \sqrt{i})\, I_0\left(\omega^* \sqrt{i}\frac{r}{R_1}\right) - I_0(\omega^* \sqrt{i})\, K_0\left(\omega^* \sqrt{i}\frac{r}{R_1}\right)}{K_0(\omega^* \sqrt{i})\, I_0(k\omega^* \sqrt{i}) - I_0(\omega^* \sqrt{i})\, K_0(k\omega^* \sqrt{i})}\right],\quad k = \frac{R_2}{R_1}. \qquad (7\text{-}25)$$

b) Boundary conditions of the first and third kind

$$T(R_1, \tau) = 0; \quad \frac{\partial T(R_2, \tau)}{\partial r} = h\{\operatorname{Im} Ae^{i(\omega\tau+\varepsilon)} - T(R_2, \tau)\}.$$

Temperature function

$$T(r, \tau) = \operatorname{Im}\Bigg[Ae^{i(\omega\tau+\varepsilon)} \times$$

$$\times \frac{K_0(\omega^* \sqrt{i})\, I_0\left(\omega^* \sqrt{i}\frac{r}{R_1}\right) - I_0(\omega^* \sqrt{i})\, K_0\left(\omega^* \sqrt{i}\frac{r}{R_1}\right)}{\Delta}\Bigg], \tag{7-26}$$

where

$$\Delta = K_0(\omega^* \sqrt{i})\, I_0(k\omega^* \sqrt{i}) +$$

$$+ \frac{m\sqrt{i}}{Bi_2}[K_0(\omega^* \sqrt{i})\, I_1(k\omega^* \sqrt{i}) + I_0(\omega^* \sqrt{i})\, K_1(k\omega^* \sqrt{i})]; \quad Bi_2 = h_2 R_1.$$

7-2. Calculations of Temperature Fields of Concrete Dams During the Period of Use

Concrete dams have a complex geometric shape. Calculations of the spatial temperature fields in such structures represent significant computational difficulties. We therefore usually limit ourselves to analysis of the planar temperature fields in the vertical transverse cross sections of a dam.

In this section, based on the method of finite differences, we shall present calculation plans considering the outline of the profile of the dam, the boundary conditions on its surface, the influence of the base, etc. These calculation plans concern massive gravity, arch-gravity and arch dams.

Basic Statements of the Calculation Method

A diagram of a typical planar area is presented in Figure 7-1[1].

[1]This area will be used to calculate the temperature field of a blind section of the Ust'-Ilimsakay Hydroelectric Power Plant Dam.

The base beneath the dam is replaced by the rectangle ABMN, the dimensions of which are selected such that the thermal perturbation from the bottom of the dam does not reach its lower or lateral boundaries.

In accordance with this, on the lines AB and MN we assume

$$\frac{\partial T}{\partial x}=0,$$

while on the lower boundary AN we assign either boundary conditions of the second kind

$$\frac{\partial T}{\partial z}=-\frac{1}{\lambda}\eta_{r},$$

where η_r is the geothermal heat flux from the earth, or boundary conditions of the first kind

$$T(0, x, \tau)=T_H=\text{const}.$$

The geothermal flux η_r averages 0.02-0.04 kcal/(m$^2\cdot$hr), the temperature at great depths T_H is established as a result of preliminary thermal studies of the rock in the region of the dam.

In the sectors wet with water CD on the upstream side and PL on the downstream side, the surface temperature of the dam and base are assumed equal to the water temperature.

In the general case, in the sector in which the dam contacts the air (line DEFGHKL), boundary conditions of the third kind should be assigned.

For bodies of complex outline, the use of boundary conditions of the third kind hinders calculation.

However, we can recommend an approach consisting in replacement of boundary conditions of the third kind with boundary conditions of the first kind, which simplifies calculation for determination of temperature fields of concrete dams during the period of use, without leading to significant distortion of the end results.

Let us assume the ambient temperature changes according to the rule

$$\psi(\tau)=T_c+A\sin(\omega\tau+\varepsilon), \qquad (7\text{-}27)$$

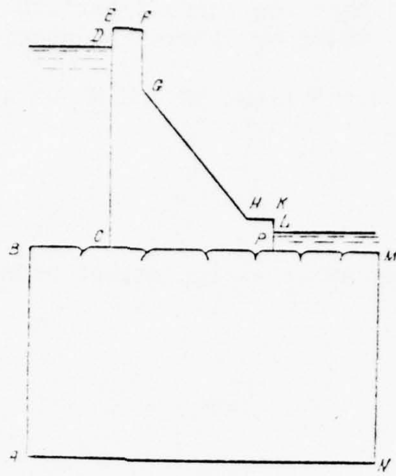


Figure 7-1. Vertical Transverse Blind Section of Concrete Gravity Dam

where T_c is the mean annual temperature; A and ω are the amplitude and frequency of oscillations of temperature; ε is the initial phase.

Under these conditions, the surface temperature of the body in the quasistable mode will be T_π, which can be represented as

$$T_{\text{п}} = T_{\text{п.ст}} + T_{\text{п.q.ст}},$$

where $T_{\pi.CT}$ is the stable component of the temperature, resulting from term T_c in expression (7-27); $T_{\pi.q.CT}$ is the quasistable harmonic component of temperature.

Due to the "massiveness" of concrete dams, in determining $T_{\pi.q.CT}$ we can use the formula for calculation of the temperature of a semilimited body (formula (7-12)), assuming in it x = 0.

Thus, we produce:

$$T_{\text{п.q.ст}} = \frac{hA}{\sqrt{k^2 + (k+h)^2}} \sin(\omega\tau + \varepsilon - \delta),$$

where

$$h = \frac{\alpha}{\lambda};\ k = \sqrt{\frac{\omega}{2a}};\quad \delta = \operatorname{arctan}\frac{k}{k+h}.$$

As was noted in § 7-1, even with simple replacement of the surface temperature of a semilimited body with the air temperature, an error of about 3% is made.

In order to determine the stable component of temperature at the surface of the mass $T_{\pi.CT}$, let us introduce an additional layer of concrete and on its surface assign temperature T_c. The thickness of the layer R_0 will be calculated by the formula

$$R_0 = \lambda \frac{1}{\alpha_e},$$

where $\alpha_e = \dfrac{1}{\dfrac{1}{\alpha} + \dfrac{R_{o\pi}}{\lambda_{o\pi}}}$ is the effective heat transfer coefficient; α is the heat transfer coefficient to the air; $\lambda_{o\pi}$ and $R_{o\pi}$ are the heat transfer coefficients and thickness of the deck.

The thickness of the additional layer R_0 is usually not great. Thus, when there is no heat insulation R_0 = 0.1 m, where α_e = 2 kcal/(m^2·hr·C) the thickness of the layer is R_0 = 1 m.

The correctness of this approach increases with increasing "massiveness" of the structure and also as its thermal state approaches the quasistable state.

On the calculation area of the dam and base, using the system of straight lines $z = ih_z^{(s)}$, $x = jh_x^{(s)}$ ($i = 0, 1, \ldots, n^{(s)}$; $j = 0, 1, \ldots, m^{(s)}$) we enter a rectangular grid. To the base we assign the index s = 0, to the dam, s = 1.

Steps $h_x^{(s)}$ and $h_z^{(s)}$ are selected such that the boundary junctions of the difference grid either lie precisely on the boundary curve of the calculation area or, at least, as close as possible to it. Furthermore, assuming $h_x^{(0)} = h_x^{(1)}$, we can match the difference grids for the two areas at the line of interface of the dam and the base.

We shall call a junction regular if the neighboring junctions, forming a five-point cross
$$\begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix}$$
are located either within the calculation area or on its boundary.

The points of intersection of the lines forming the grid with the boundary of the area will be called boundary points.

Thus, in Figure 7-2, points 1 and 2 are regular, point 0 is irregular, points 3, 4 and 5 are boundary points.

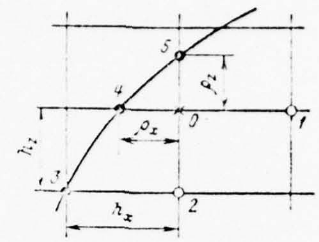


Figure 7-2. Diagram of Placement of Boundary Points on Contour of Body

In order to approximate the system of differential equations of heat conductivity for the dam and base, we utilize an explicit six-point plan.

Then the temperature at the regular junctions of the difference grid $T_{i,j,k+1}$ at moment in time $\tau_{k+1} = (k + 1)\ell$ ($k = 0, 1, \ldots$; ℓ is the time step) can be calculated by the algorithm

$$T^{(s)}_{i,j,k+1} = A_s T^{(s)}_{i,j,k} + M^{(s)}_z (T^{(s)}_{i+1,j,k} + T^{(s)}_{i-1,j,k}) + \\ + M^{(s)}_x (T_{i,j+1,k} + T_{i,j-1,k}) \ (i=0,\ 1, \ldots,\ n^{(s)}; \\ j=0,\ 1, \ldots,\ m^{(s)};\ k=0,\ 1 \ldots;\ s=0,\ 1), \tag{7-28}$$

where

$$A_s = 1 - 2M^{(s)}_z - 2M^{(s)}_x;$$
$$M^{(s)}_x = \frac{a_s l}{h^2_x}; \quad M^{(s)}_z = \frac{a_s l}{(h^{(s)}_z)^2}.$$

The algorithms for calculation of temperature at the boundaries of the calculation area are:

a) The lower boundary of the base (line AN) in Figure 7-1 with boundary conditions of the second kind

$$T^{(0)}_{0,j,k+1} = A_0 T^{(0)}_{0,j,k} + M^{(0)}_x (T^{(0)}_{0,j+1,k} + T^{(0)}_{0,j-1,k}) + 2M^{(0)}_z T^{(0)}_{1,j,k} + + 2M^{(0)}_z \frac{h^{(0)}_z}{\lambda} \eta_r; \tag{7-29}$$

b) The lateral boundaries of the base AB and MN:

Line AB

$$T^{(0)}_{i,0,k+1} = A_0 T^{(0)}_{i,0,k} + M^{(0)}_z (T^{(0)}_{i+1,0,k} + T^{(0)}_{i-1,0,k}) + 2M^{(0)}_x T^{(0)}_{i,1,k}; \tag{7-30}$$

Line MN

$$T^{(0)}_{i,m^{(0)},k+1} = A_0 T^{(0)}_{i,m^{(0)},k} + M_z (T^{(0)}_{i+1,m^{(0)},k} + T^{(0)}_{i-1,m^{(0)},k}) + + 2M^{(0)}_x T^{(0)}_{i,m^{(0)}-1,k}. \tag{7-31}$$

c) Line CP -- the interface between the dam and the base

$$T_{i,j,k+1} = AT_{i,j,k} + G_1 T_{i+1,j,k} + G_0 T_{i-1,j,k} + + E(T_{i,j+1,k} + T_{i,j-1,k}), \tag{7-32}$$

where

$$A = \frac{1}{2}[h^{(0)}_z c_0 \gamma_0 (1 - M^{(0)}_z - 2M^{(0)}_x) + h^{(1)}_z c_1 \gamma_1 (1 - M^{(1)}_z - 2M^{(1)}_x)];$$

$$E = \frac{1}{2}\left[\frac{h^{(0)}_z}{h_x}\frac{\lambda_0 l}{h_x} + \frac{h^{(1)}_z}{h_x}\frac{\lambda_1 l}{h_x}\right]; \; G_s = \frac{\lambda_s l}{h^{(s)}_z} \; (s = 0, 1).$$

d) Lines BC and PM (upper boundary of base), line CDEFGHKLP (profile of dam).

As was noted earlier, boundary conditions of the first kind are assigned on these lines.

At the boundary junctions (for example at junction 3 in Figure 7-2) in each time step we enter the temperature of the water or the corresponding converted air temperature.

The situation is somewhat more complex in the case of irregular junctions.

In the literature, several methods have been suggested for determining the temperature at irregular junctions. Let us stop to discuss two of these.

1. Simple transfer of the boundary temperature. The values of temperature at irregular junctions are assumed equal to the values of temperature at the nearest boundary points. For example, the temperature at point 0 (Figure 7-2) is assumed equal to the temperature at junction 4 or 5.

In this case, the error in approximation in irregular junctions may be on the order of O(h), whereas at regular junctions its order is $O(h_2 + \ell)$.

2. Linear interpolation of the boundary temperature. For definition, let us refer to Figure 7-2.

The temperature at irregular point 0 is assumed equal to

$$T_0 = \frac{\rho_x T_1 + h_x T_4}{h_x + \rho_x} \tag{7-33}$$

or

$$T_0 = \frac{\rho_z T_2 + h_z T_5}{h_z + \rho_z}. \tag{7-34}$$

The values of ρ_x, ρ_z, h_x and h_z are obvious from Figure 7-2, T_0, T_1, T_2, T_4 and T_5 are the temperatures at points 0, 1, 2, 4 and 5.

Equations (7-33) and (7-34) are stable with all h and ℓ and have approximation error on the order of $O(h^2)$.

The algorithms presented provide for stable computation if the stability condition is fulfilled:

$$\min_s l_s \leqslant \frac{1}{2a_s\left[\frac{1}{(h_1^{(s)})^2} + \frac{1}{(h_2^{(s)})^2}\right]}.$$

Based on the relationships presented, a program was created -- a method of computer calculation of the temperature field of concrete dams during the period of operation. The program allows calculation of the temperature field of concrete dams of any profile.

The algorithm was written so that using the selected steps of the coordinates, a rectangular grid is formed, which fully covers the rectangle modeling the base and sufficiently closely covers the dam profile. The contour of the profile is assigned in the initial information by the vertical coordinates of the junctions of the calculation grid. The vertical steps are generally different for the areas of the dam and base; the horizontal steps are identical, allowing the junctions of the calculation grid to coincide at the interface between the dam and the base.

Otherwise, the calculation program of temperature fields in concrete masses during the period of operation is similar to that described in § 5-4 for calculation of temperature fields of concrete masses during the period of construction.

The method developed and computer calculation program created allow us to determine the temperature field of concrete dams during the period of operation with any initial distribution of temperature.

As was noted in the beginning of this section, in structures under the periodic influence of external factors (ambient temperature, etc.), a quasistable state is achieved. The time of onset of the quasistable temperature state is established by comparing the results of calculation of temperature for two or three annual cycles. If the values of temperature are quite similar at corresponding moments in time and corresponding points in the structure, the temperature field during one of the annual cycles compared is taken as a quasistable field. This quasistable temperature field is totally independent of the initial temperature state of the structure.

Temperature Fields of Certain Concrete Dams During the Period of Use

1. Ust'-Ilimskaya Hydroelectric Power Plant Concrete Dam (Plan Version)

The region of construction of the Ust'-Ilimskaya Power Plant is distinguished by sharp continental climate. The mean annual air temperature is -3.9 C. During the 120 days of the winter season, the mean temperature around the clock is lower than -15 C, during 95 days of the winter it is lower than -20 C. The difference between the maximum summer and minimum winter temperatures of the air reaches 94 C.

Data on the mean monthly air temperatures in the region of the Ust'-Ilimskaya Power Plant and the expected thermal conditions in the reservoir are presented in § 2-3.

In order to determine the time required for onset of the quasistable mode, results were compared from calculation of the temperature field of the dam for 15 January and 15 July after 20, 40 and 60 years from the beginning of calculation. Comparison of these data showed that on 15 January, the maximum temperature difference calculated at corresponding points after 20 and 40 years was 0.14 C, on 15 January after 40 and 60 years -- 0.027 C.

On this basis, in all of the data presented below concerning the temperature state of the Ust'-Ilimskaya Dam, the calculation period of onset of quasistable mode is taken as 40 years.

Obviously, the calculated time of onset of the quasistable mode depends both on the corresponding characteristics of the structure itself and on factors acting upon it, as well as the initial thermal state of the object assumed in the calculations.

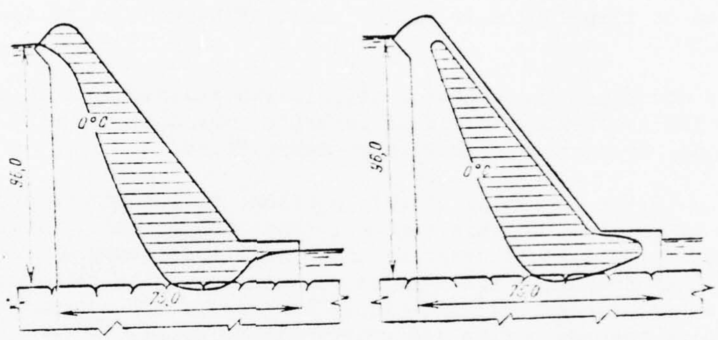


Figure 7-3. Quasistable Temperature Field of Spillway Section of Ust'-Ilimskaya Hydroelectric Power Plant Dam. a, On 15 January; b, On 15 July.

Some of the results of calculation of the quasistable temperature field of the concrete dam at Ust'-Ilimskaya are presented in Figure 7-3.

The zero isotherm passes through the center of the width of the profile of the spillway dam, from its crest to the base, descending into the rock to a depth of 2-2.5 m, being deflected downstream, then returns to the dam approximately at a distance of 1/4 of its width at the base from the downstream wedge (Figure 7-3). Then the zero isotherm departs the dam at the level of the downstream water. During the summer, due to seasonal variations in air temperature, the downstream face of the dam thaws to a depth of 3-4 m.

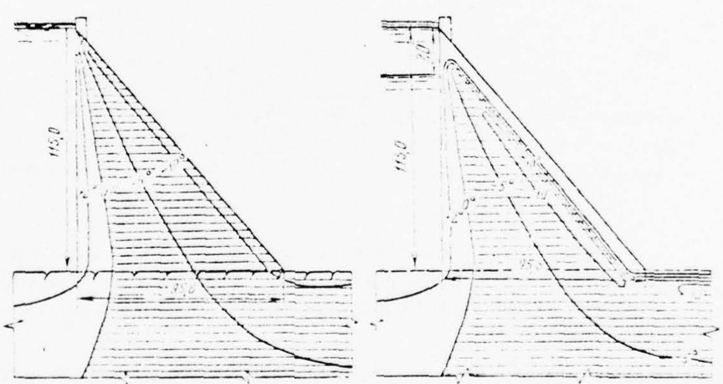
The zone of below-freezing temperatures in the base of a blind dam is located 2-2.5 m closer to the upstream side in comparison to the corresponding sectors of a spillway dam.

The base on the downstream side is frozen to a considerable depth. The crest of the dam is under unfavorable temperature conditions: it almost completely freezes in the winter, and a significant portion of its volume thaws in the summer.

2. The Concrete Dam of the Kolymskaya Power Plant (Plan Version)

The climatic conditions of the region of construction are severe: the mean annual air temperature is -12.0 C, the mean temperature in January is -38 C; over 90 days per year there is a mean daily temperature of -30 C or lower. The dam is located in an area where the ground is permanently frozen ("permafrost"). The thickness of the permafrost is 100-200 m.

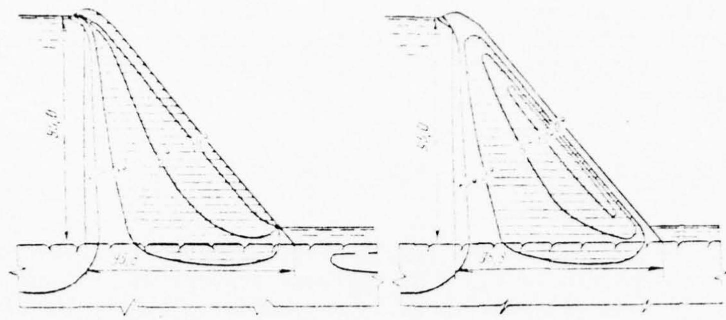
According to preliminary data supplied by the planning organization, in the Kolyma Reservoir, the greatest changes in water temperature will be observed in the surface layer down to a depth of 30-35 m. In summer, with average weather conditions at the surface (down to a depth of 2-3 m) the water temperature will be about 14-15 C. Lower, down to a depth of 30-35 m, the water temperature will fall rapidly and at the lower face of this layer will reach 5-6 C. In the bottom layer (from 35 m to the bottom), the water temperature will decrease quite smoothly, down to about 4 C at the bottom.



a) b)

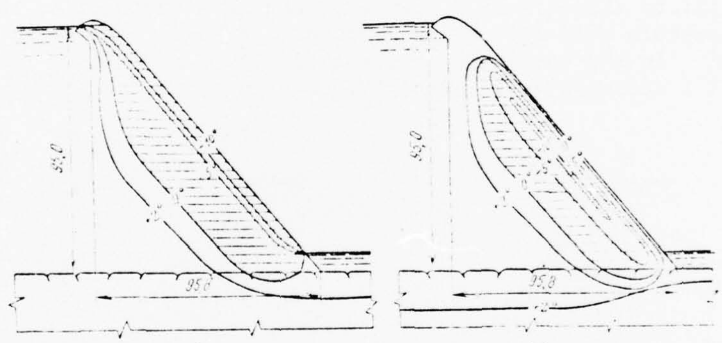
Figure 7-4. Quasistable Temperature Field of Blind Section of Concrete Dam of Kolyma Power Plant (Plan Version). a, 15 January; b, 15 July

In the winter (beneath the ice cover) down to a depth of 30-35 m the water temperature will rise from 0 to 2-3 C, and in the lower layers will be practically constant at 3 C.



a) b)

Figure 7-5. Quasistable Temperature Field of Spillway Section of Kolyma Power Plant Dam (Plan Version). a, 15 January; b, 15 July



a) b)

Figure 7-6. Transient Temperature Mode Through Cross Section of Kolyma Power Plant Dam (Plan Version) 20 Years after Construction. a, 15 January; b, 15 July

These water and air temperatures were used to calculate the temperature mode of spillway and blind sections of the dam.

As we can see from Figures 7-4 and 7-5, during the period of use, 3/4 of the thickness of the profile of the spillway portion of the dam will be frozen. During the summer months, the concrete on the downstream face of the dam will thaw to a depth of 2-3 m. At the base, the zone of below-freezing temperatures will extend to a depth of 6-9 m. The remaining portion of the base will be at temperatures above freezing.

The transient temperature mode of the structure is illustrated by the data of Figure 7-6. The isotherms are constructed for moments in time 20 years (Figure 7-6) and 50 years (Figure 7-5) after completion of construction. It was assumed in the calculations that the initial temperature of the structure was +25 C, of the base + 1.0 C. In the base of those sections of the spillway dam located in the stream bed portion of the line, the thawed ground freezes at the base of the structure to a depth of up to 6-9 m. The base in the remaining portions of the spillway portion of the dam will be gradually converted over a period of 10-15 years from permafrost to the thawed state, the zone of below-freezing temperatures in the structure will remain in contact with the permafrost zone of the base for a long period of time. The sections of the blind shoreline dam in the quasistable state are in various conditions: beneath sections located at the upper levels at the sides of the canyon, the base is fully frozen; beneath sections located closer to the power plant section, the base is only partially frozen.

CHAPTER 8. METHODS OF CALCULATION OF THE AIR TEMPERATURE IN CLOSED CAVITIES IN DAMS

8-1. Statement and Solution of the Basic Problem

In analysis of the thermal mode of massive counterforce and counterforce dams, as well as dams with expanded seams (such as the Bratsk Power Plant), we frequently must calculate the temperature fields of masses containing closed air cavities of various sizes and shapes. Precise solution of these problems is not at present possible.

Therefore, the general solution of the problem is usually divided into two stages. In the first stage, the air temperature in the cavity is determined; in the second stage, these results, together with the remaining necessary information, are used as initial data to calculate the temperature field of the concrete mass.

Calculations in the second stage can be performed by methods outlined in previous chapters.

Therefore, primary attention in this chapter will be given to methods of computer calculation of the air temperature in closed cavities in dams developed by the All-Union Scientific Research Institute for Hydrography imeni B. Ye. Vedeneyev [101], allowing us to consider most of the factors which have an influence under the actual conditions of construction and operation of dams.

The essence of the methods suggested consists in the following.

In the three-dimensional structure, an area is set aside, including a solid body and a cavity. The solid body is divided into simple elements, within the limits of which the temperature field is assumed even.

The temperature curve of the surrounding medium is approximated by an exponential function with a constant time step $\Delta\tau = \tau_{\ell} - \tau_{\ell-1}$ ($\ell = 1, 2, \ldots;$ $\tau_0 = 0$). It is assumed that in any time step the air temperature in the cavity is also constant.

At the end of each time step, a thermal balance equation is composed for the dam, the solution of the equation is the temperature of the cavity during the time interval from τ to $\tau + \Delta\tau$.

The change in temperature in the cavity is defined as the combination of these solutions at various moments in time.

Suppose the three-dimensional structure here being studied contains a cavity surrounded by m + 1 elements j (j = 0, 1, 2,..., m). Among these elements we will distinguish the base (j = 0), flat walls (j = 1, 2,..., p) and curved walls (sectors of hollow cylinders) (j = p + 1, p + 2,..., m)[1].

At the boundaries of flat and curved walls, we assign the most general boundary conditions of the third kind. The base is replaced by a plate of sufficient thickness for which on the surface turned toward the cavity we can assign boundary conditions of the third kind, while on the outer surface we assign boundary conditions of the second kind, considering the geothermal flux from the earth, or boundary conditions of the first kind with constant temperature.

Under these conditions, the problem of heat conductivity for flat and curved walls in time step $\Delta\tau = \tau_\ell - \tau_{\ell-1}$ (ℓ = 1, 2,...) can be formulated as follows[2]

$$\frac{\partial T}{\partial \tau} = a \frac{1}{\xi^i} \frac{\partial}{\partial \xi}\left(\xi^i \frac{\partial T}{\partial \xi}\right) \quad (R_1 < \xi < R_2,\ \tau_{l-1} < \tau < \tau_l;\ l = 1, 2\ldots;\ i = 0 \vee 1); \tag{8-1}$$

$$T(\xi, \tau_{l-1}) = f(\xi) \quad (R_1 \leqslant \xi \leqslant R_2); \tag{8-2}$$

$$\frac{\partial T(R_1, \tau)}{\partial \xi} = -h_1[T_1 - T(R_1, \tau)];$$

$$\frac{\partial T(R_2, \tau)}{\partial \xi} = h_2[T_c - T(R_2, \tau)]. \tag{8-3}$$

Here T_c is the temperature of the air outside; T_1 is the desired temperature in the cavity; $f(\xi)$ is the temperature established in the wall at the end of the time step previous to the step in question; a and λ are the coefficients of temperature conductivity and heat conductivity of the wall material; h_1 and h_2 are the relative heat transfer coefficients at the internal (turned toward the cavity) and external surfaces of the wall.

For flat walls: $\xi = x$, $i = 0$, $R_1 = 0$, $R_2 = R$; for curved walls: $\xi = r$, $i = 1$.

The statement of the problem for the base is similar to the above, except that we must assume that, as for a flat wall, $\xi = x$, $i = 0$, $R_1 = 0$, $R_2 = R$,

---

[1]Henceforth, the subscript 0 will be used to mark everything related to the base, the subscripts 1, 2,... -- everything relating to the walls.

[2]For simplicity, we omit the subscript j.

while the second boundary condition on the exterior surface x = R is replaced by the condition

$$\frac{\partial T(R,\tau)}{\partial x} = \frac{1}{\lambda}\eta_r, \tag{8-4}$$

where η_r is the geothermal heat flux, or by the condition

$$T(R,\tau) = T_{\text{п}}. \tag{8-5}$$

In the general case, the heat-physical characteristics a and λ, relative heat transfer coefficients h_1 and h_2, geothermal flux η_r, temperatures T_1 and T_c are functions of time, but within the limits of each step they are assumed constant.

The solution of the problem (8-1)-(8-3) was produced earlier (see Chapter 4):

$$T = \Phi(\xi) - \sum_{n=1}^{\infty} \frac{\bar{\Phi}_n - \bar{f}_n}{\| U_0 \|^2} U_0\left(\mu_n \frac{\xi}{R}\right) e^{-\mu_n^2 \frac{a(\tau-\tau_{i-1})}{R^2}}, \tag{8-6}$$

where $\Phi(\xi)$ is a substitution function; $U_0(\mu_n \frac{\xi}{R})$ is the Eigenfunction of the problem; μ_n is the root of the characteristic equation; R is the characteristic dimension of the body;

$\| U_0 \|^2 = \int_{R_1}^{R_2} \xi^{\nu} U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$ — is the square of the norm of the Eigenfunction;

$$\bar{\Phi}_n = \int_{R_1}^{R_2} \xi^{\nu} \Phi(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$

$$\bar{f}_n = \int_{R_1}^{R_2} \xi^{\nu} f(\xi) U_0\left(\mu_n \frac{\xi}{R}\right) d\xi.$$

For flat and curved walls with boundary conditions of the third kind, the substitution function $\Phi(\xi)$ can be represented as

$$\Phi(\xi) = T_1 + (T_c - T_1) F_{Ia}(\xi),$$

where $F_{Ia}(\xi)$ is a F function of the first kind, satisfying the differential equation

$$\nabla^2 F_{Ia}(\xi) = \frac{1}{\xi^\nu} \frac{d}{d\xi} \left(\xi^\nu \frac{dF_{Ia}}{d\xi} \right) = 0 \tag{8-7}$$

and the boundary conditions

$$\frac{dF_{Ia}(R_1)}{d\xi} = h_1 F_{Ia}(R_1); \qquad \frac{dF_{Ia}(R_2)}{d\xi} = h_2 [1 - F_{Ia}(R_2)]. \tag{8-8}$$

For a slab base with boundary conditions of the second kind assigned on the outer surface $\xi = R_2 = R$

$$\Phi(x) = T_1 + \frac{q_c}{\lambda} F_{Ia}(x),$$

where $F_{Ia}(x)$ satisfies equation (8-7) with the boundary conditions

$$\frac{dF_{Ia}(0)}{dx} = h_1 F_{Ia}(R_1); \qquad \frac{dF_{Ia}(R)}{dx} = 1. \tag{8-9}$$

If a condition of the first kind is assigned on the outer surface of the slab base, then

$$\Phi(x) = T_1 + (T_n - T_1) F_{Ia}(x), \tag{8-10}$$

where $F_{Ia}(x)$ satisfies equation (8-3) and the boundary conditions

$$\frac{dF_{Ia}(R_1)}{dx} = h_1 F_{Ia}(R_1); \qquad F_{Ia}(R_2) = 1. \tag{8-11}$$

Thus, for walls and slab bases with boundary conditions (8-3) or (8-5):

$$T = T_1 + T_c F_{1a}(\xi) + \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right)\left[\bar{f}_n - T_1 (N_n + \right.$$

$$\left. + \bar{F}_{1a}) - T_c \bar{F}_{1a}\right] \exp\left[-\mu_n^2 \frac{a(\tau - \tau_{i-1})}{R^2}\right]; \qquad (8\text{-}12)$$

for a slab base with boundary conditions (8-3) (first condition) and (8-4)

$$T = T_1 + \frac{1}{\lambda} \eta_r F_{1a}(x) + \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{x}{R}\right)\left[\bar{f}_n - T_1 N_n - \right.$$

$$\left. - \frac{1}{\lambda} \eta_r \bar{F}_{1a}\right] \exp\left[-\mu_n^2 \frac{a(\tau' - \tau_{i-1})}{R^2}\right]. \qquad (8\text{-}13)$$

Here

$$N_n = \int_{R_1}^{R_2} \xi^i U_0\left(\mu_n \frac{\xi}{R}\right) d\xi;$$

$$\bar{F}_{1a} = \int_{R_1}^{R_2} \xi^i F_{1a}(\xi) U_0\left(\mu_n \frac{R}{\xi}\right) d\xi.$$

The thermal balance equation for the cavity at the end of time step $\Delta\tau$ can be written as:

$$-\sum_{j=0}^{m} (\lambda S G)_j = Q_1 + Q_2 + Q_3. \qquad (8\text{-}14)$$

The terms in the left portion of expression (8-14) define the quantity of heat leaving the cavity during $\Delta\tau$ through the surfaces of the elements surrounding the cavity, summation being conducted with respect to all j of the elements, where

$$G = \int_0^{\Delta\tau} \left(\frac{\partial T}{\partial \xi}\right)_{\xi = R_1} d\tau,$$

S is the area of the surface of the jth element; λ is the heat conductivity factor of the material of the jth element. The terms in the right portion of expression (8-14) determine the quantity of heat liberated in the cavity due to the effect of additional power supplies (heaters, etc.) (Q_1), due to condensation (evaporation) of moisture (Q_2), and also entering the cavity from open water surfaces (Q_3).

In composing the thermal balance, the heat content of the air in the cavity is ignored. Furthermore, it is assumed that the intensity of heat fluxes $q_r = dQ_r/d\tau$ ($r = 1, 2, 3$) is constant during a time step $\Delta\tau$, i.e.

$$Q_r = q_r \Delta\tau.$$

This means that q_r, which is generally dependent on time, is approximated by piecewise-constant functions of τ.

The quantity of heat Q_3 entering the cavity from open water surfaces is established in accordance with Newton's law

$$Q_3 = \alpha_B (T_B - T_1) S_B \Delta\tau,$$

where α_B is the heat transfer factor from the water surface to the air; T_B is the temperature of the water surface, the area of which is S_B.

Considering the conditions outlined above, the air temperature in cavity T_1 during time step $\Delta\tau$ ($\tau_{\ell-1} < \tau < \tau_\ell$ ($\ell = 1, 2, \ldots$) is equal to:

$$\begin{aligned} T_1 = [-\{\lambda SRA\}_0 + \sum_{j=1}^{m} \{\lambda SR(B - A)\}_j + \alpha_B S_B \Delta\tau]^{-1} [(q_1 + \\ + q_2 + \alpha_B S_B T_B) \Delta\tau + \eta_r \{S [F'(R_1) \Delta\tau + RB]\}_0 - \{\lambda SRC\}_0 + \\ + \sum_{j=1}^{m} \{\lambda ST_c [F'(R_1) \Delta\tau + RB]\}_j - \sum_{j=1}^{m} \{\lambda SRC\}_j]. \end{aligned} \tag{8-15}$$

Here

$$A = \frac{1}{a} \sum_{n=1}^{\infty} \frac{N_n}{\mu_n \| U_0 \|^2} U_1 \left(\mu_n \frac{R_1}{R} \right) \left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}} \right);$$

$$B = \frac{1}{a} \sum_{n=1}^{\infty} \frac{F_{1n}}{\mu_n \| U_0 \|^2} U_1 \left(\mu_n \frac{R_1}{R} \right) \left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}} \right);$$

$$C = \frac{1}{a} \sum_{n=1}^{\infty} \frac{f_n}{\mu_n \| U_0 \|^2} U_1 \left(\mu_n \frac{R_1}{R} \right) \left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}} \right);$$

$U_1(\mu_n \frac{\xi}{R})$ is determined from the formula

$$\frac{d}{d\xi} U_0\left(\mu_n \frac{\xi}{R}\right) = -\frac{\mu_n}{R} U_1\left(\mu_n \frac{\xi}{R}\right);$$

$$F'(R_1) = \left.\frac{dF(\xi)}{d\xi}\right|_{\xi=R_1}.$$

The braces contain terms relating to the jth element. Summation is conducted with respect to elements $j = 1, 2, \ldots, m$, with the exception of the plate base, for which $j = 0$.

If at the outer surface of the plate base we assign boundary conditions of the first kind (8-5), we need not distinguish the plate base and perform summation with respect to all j ($j = 0, 1, \ldots, m$); in algorithm (8-15), the braces with subscript 0 should be assumed equal to 0 and for the plate base ($j = 0$), T_c should be replaced by T_H.

If the surface of the base is fully covered with water, then in expression (8-15), summation is performed with respect to $j = 1, 2, \ldots, m$, and we must assume $\{\ \}_0 = 0$.

In calculating the air temperature in the cavity, the initial distribution of temperature in each element $f(\xi)$ in which, figuratively speaking, the prehistory of the process is reflected must be considered.

For simplification of computation in each time step function $f(\xi)$ is approximated by an sth power polynomial

$$f(\xi) = \sum_{\rho=0}^{s} b_\rho \xi^\rho,$$

where the coefficient b_ρ is defined on the basis of the values of temperature at moment in time $\tau = \tau_{\ell-1}$ at $(s + 1)$ equally spaced points. Calculation of the temperature through the cross section of the element is conducted from step to step (beginning with $\tau = 0$) by formulas (8-12) or (8-13); the values of b_ρ are established as a result of solution of the corresponding systems of algebraic equations.

Then

$$\bar{f}_n = \sum_{\rho=0}^{s} b_\rho I_\rho^{(n)},$$

where

$$I_{\rho}^{(n)} = \int_{R_1}^{R_2} \xi^{\rho} U_0\left(\mu_n \frac{\xi}{R}\right) d\xi.$$

As will be shown below, the values of the integrals $I_{\rho}^{(n)}$ are determined from the recurrent relationships.

8-2. Method of Calculation of Air Temperature in the Cavities in a Dam with Flat Covers

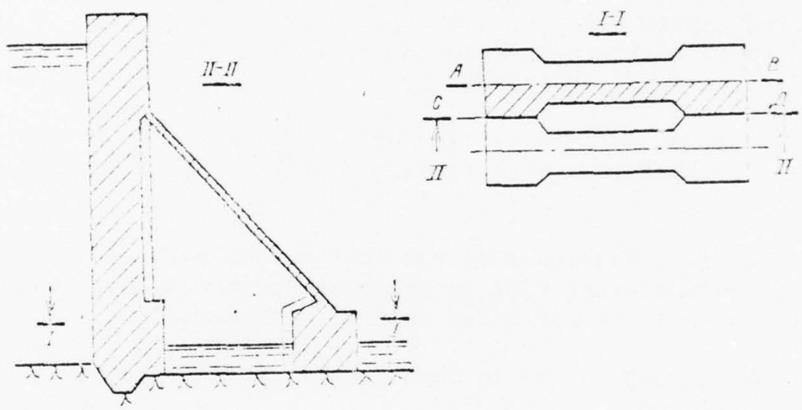


Figure 8-1. Cross Section of Sections with Cavities

Figure 8-1 shows a cross section of a section with a cavity located in the central portion of the dam. It is assumed that the neighboring sections of the dam are under identical temperature conditions. Therefore, we can use the shaded area in Figure 8-1 as our calculation area (lines AB and CD are the axes of symmetry). On the axes of symmetry, homogeneous boundary conditions of the second kind are established:

$$\frac{\partial T(R,\tau)}{\partial x} = 0.$$

The cavity is limited by flat walls and the base plate.

For the flat walls

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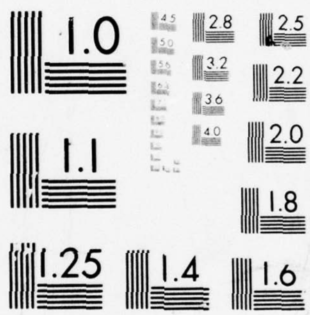

MICROCOPY RESOLUTION TEST CHART

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$$\xi = x,\ i = 0,\ R_1 = 0,\ R_2 = R,\ 0 < x < R;$$

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R}; \quad \mathrm{Bi}_1 = h_1 R;$$

$$U_1\left(\mu_n \frac{x}{R}\right) = \mu_n \sin \mu_n \frac{x}{R} - \mathrm{Bi}_1 \cos \mu_n \frac{x}{R};$$

μ_n is the root of the characteristic equation

$$\cot \mu_n = \frac{\mu_n^2 - \mathrm{Bi}_1 \mathrm{Bi}_2}{\mu_n (\mathrm{Bi}_1 + \mathrm{Bi}_2)}; \quad \mathrm{Bi}_2 = h_2 R;$$

$$\| U_0 \|^2 = \frac{R}{2}\left[\mathrm{Bi}_1^2 + \mathrm{Bi}_1 + \mu_n^2 + \frac{\mathrm{Bi}_2 (\mathrm{Bi}_1^2 + \mu_n^2)}{\mathrm{Bi}_2^2 + \mu_n^2}\right];$$

$$N_n = \frac{R}{\mu_n}\left[\mathrm{Bi}_1 + (-1)^{n+1} \mathrm{Bi}_2 \sqrt{\frac{\mathrm{Bi}_1^2 + \mu_n^2}{\mathrm{Bi}_2^2 + \mu_n^2}}\right];$$

$$F_{1a}(x) = \frac{\mathrm{Bi}_2}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2}\left(1 + \mathrm{Bi}_1 \frac{x}{R}\right);$$

$$F'_{1a}(R_1) = \left.\frac{dF_{1a}(x)}{dx}\right|_{x=R_1=0} = \frac{\mathrm{Bi}_1}{R} \frac{\mathrm{Bi}_2}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2};$$

$$\bar{F}_{1a} = N_n - \frac{\mathrm{Bi}_1 R}{\mu_n};$$

$$U_1\left(\mu_n \frac{R_1}{R}\right) = U_1(0) = -\mathrm{Bi}_1.$$

For the plate base

$$U_0\left(\mu_n \frac{x}{R}\right) = \mu_n \cos \mu_n \frac{x}{R} + \mathrm{Bi}_1 \sin \mu_n \frac{x}{R}; \quad \mathrm{Bi}_1 = h_1 R;$$

$$U_1\left(\mu_n \frac{x}{R}\right) = \mu_n \sin \mu_n \frac{x}{R} - \mathrm{Bi}_1 \cos \mu_n \frac{x}{R};$$

μ_n is the root of the characteristic equation

$$\cot \mu_n = \frac{\mu_n}{\mathrm{Bi}_1};$$

$$\| U_0 \|^2 = \frac{R}{2}(\mathrm{Bi}_1^2 + \mathrm{Bi}_1 + \mu_n^2); \quad N_n = \frac{R\,\mathrm{Bi}_1}{\mu_n};$$

$$F'_{1a}(x) = R\left(\frac{1}{\mathrm{Bi}_1} + \frac{x}{R}\right); \quad \bar{F}_{1a} = \frac{R^2}{\mu_n^2}(-1)^{n+1} \sqrt{\mathrm{Bi}_1^2 + \mu_n^2};$$

$$F'_{1a}(R_1) = 1, \quad U_1\left(\mu_n \frac{R_1}{R}\right) = -\mathrm{Bi}_1.$$

Special calculations have shown that the initial temperature of an element surrounding the cavity in each step can be approximated by a fourth power polynomial, i.e., we can assume:

$$f(x) = \sum_{\rho=0}^{4} b_\rho x^\rho. \tag{8-16}$$

In order to determine the coefficients b_ρ ($\rho = 0, 1, \ldots, 4$), we should use values of temperature at moment $\tau = \tau_{\ell-1}$ at five equally spaced points through the cross section of the wall or base plate. The solution of the system of five algebraic equations with 5 unknowns b_ρ thus produced is not difficult.

From formula (8-16) it follows that

$$f_n = \sum_{\rho=0}^{4} b_\rho I_\rho^{(n)},$$

where

$$I_\rho^{(n)} = \int_0^R x^\rho U_0\left(\mu_n \frac{x}{R}\right) dx.$$

The values of the integrals $I_\rho^{(n)}$ are determined from the recurrent relationships:

for flat walls

$$I_\rho^{(n)} = \frac{R^2}{\mu_n^2}\left[R^{\rho-1}\mu_n(\mathrm{Bi}_2 + \rho)(-1)^{n+1} \times \right.$$
$$\left. \times \sqrt{\frac{\mathrm{Bi}_1^2 + \mu_n^2}{\mathrm{Bi}_2^2 + \mu_n^2}} - \rho(\rho - 1) I_{\rho-2}^{(n)}\right]; \quad \rho \geq 2;$$
$$I_0^{(n)} = N_n;$$
$$I_1^{(n)} = \frac{R^2}{\mu_n}\left[(\mathrm{Bi}_2 + 1)(-1)^{n+1}\sqrt{\frac{\mathrm{Bi}_1^2 + \mu_n^2}{\mathrm{Bi}_2^2 + \mu_n^2}} - 1\right];$$

for the base plate with boundary conditions of the second kind on the surface x = R

$$I_{\rho}^{(n)}=\frac{R^2}{\mu_n^2}\left[R^{\rho-1}\rho(-1)^{n+1}\sqrt{\mathrm{Bi}_1^2+\mu_n^2}-\rho(\rho-1)I_{\rho-2}^{(n)}\right];\quad \rho\geq 2;$$

$$I_0^{(n)}=N_n;$$

$$I_1^{(n)}=\frac{R^2}{\mu_n^2}\left[(-1)^{n+1}\sqrt{\mathrm{Bi}_1^2+\mu_n^2}-1\right].$$

Note. For flat walls separating cavities through which the axis of symmetry passes (see Figure 8-1) in all calculation formulas we must assume $\mathrm{Bi}_2 = 0$.

8-3. Method of Calculation of Air Temperature in Cavities of a Multiarch Dam

The horizontal cross section of a typical section with a cavity in a multiarch dam is shown in Figure 8-2. The calculation area in Figure 8-2 is shaded, lines AB and CD are the axes of symmetry.

In contrast to sections studied earlier, here among the element limiting the cavity we have a curved wall. For cylindrical arches this is a portion of a hollow cylinder limited by radial planes passing through the axis of the arch. It is assumed that the thermal contact between the arches and the counterforces is perfect.

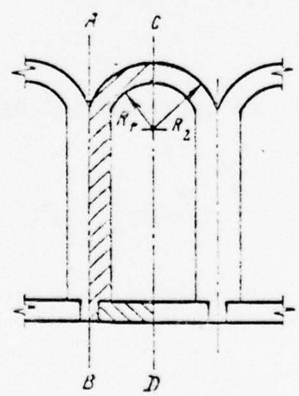

Figure 8-2. Horizontal Cross Section of Section of Multiple Arch Dam

For the curved wall

$$\xi = r;\ i = 1;\ R_1 \leqslant r \leqslant R_2,\ R_1 - \text{ is the characteristic dimension;}$$

$$U_\nu\left(\mu_n \frac{x}{R_1}\right) = \left[Y_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} Y_1(\mu_n)\right] J_\nu\left(\mu_n \frac{r}{R_1}\right) -$$
$$-\left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right] Y_\nu\left(\mu_n \frac{r}{R_1}\right);\quad \nu = 0;\ 1;\quad \mathrm{Bi}_1 = h_1 R_1;\ k = \frac{R_2}{R_1};$$

μ_n is the root of the characteristic equation

$$\mathrm{Bi}_2\{\mathrm{Bi}_1[Y_0(\mu_n) J_0(k\mu_n) - J_0(\mu_n) Y_0(k\mu_n)] + \mu_n[Y_1(\mu_n) J_0(k\mu_n) -$$
$$- J_1(\mu_n) Y_0(k\mu_n)]\} - \mu_n\{\mathrm{Bi}_1[Y_0(\mu_n) J_1(k\mu_n) - J_0(\mu_n) Y_1(k\mu_n)] +$$
$$+ \mu_n[Y_1(\mu_n) J_1(k\mu_n) - J_1(\mu_n) Y_1(k\mu_n)\} = 0;\quad \mathrm{Bi}_2 = h_2 R_1;$$

$$\| U_0 \|^2 = \frac{2R_1^2}{\pi^2 \mu_n^2} \left\{\left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-2} \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right]^2 \times\right.$$
$$\left.\times\left(1 + \frac{\mu_n^2}{\mathrm{Bi}_2^2}\right) \left(1 + \frac{\mu_n^2}{\mathrm{Bi}_1^2}\right)\right\};$$

$$N_n = \frac{2R_1^2}{\pi \mu_n^2} \left\{\left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-1} \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right] - 1\right\};$$

$$F_{1a}(r) = \frac{k\,\mathrm{Bi}_1 \mathrm{Bi}_2}{(\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2 \ln k)} \left(\frac{1}{\mathrm{Bi}_1} + \ln \frac{r}{R_1}\right);$$

$$F'_{1a}(R_1) = \frac{k\mathrm{Bi}_1 \mathrm{Bi}_2}{R_1(\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2 \ln k)};$$

$$\bar{F}_{1a} = \frac{2R_1^2 \mathrm{Bi}_2}{\pi \mu_n^2 \mathrm{Bi}_1} \left[J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)\right]^{-1} \left[J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)\right];$$

$$U_1\left(\mu_n \frac{R_1}{R}\right) = \frac{2}{\pi \mu_n}.$$

The initial temperature f(r) is approximated by the polynomial

$$f(r) = \sum_{p=0}^{4} b_{2p} r^{2p}.$$

Then

$$\bar{f}_n = \sum_{p=0}^{4} b_{2p} I_{2p}^{(n)}.$$

where the integrals

$$I_{2\rho}^{(n)} = \int_{R_1}^{R_2} r r^{2\rho} U_0\left(\mu_n \frac{r}{R_1}\right) dr$$

are calculated by the recurrent relationship

$$I_{2\rho}^{(n)} = \left(\frac{R_1}{\mu_n}\right)^{2\rho-1} \Big\{ (k\mu_n)^{2\rho-1} U_0(k\mu_n)\,[k\mathrm{Bi}_2 - (2\rho - 1)] + \\ + \mu_n^{2\rho-1} \frac{2(1-\mathrm{Bi}_1)}{\pi \mathrm{Bi}_1} + (2\rho-1)^2 \left(\frac{\mu_n}{R_1}\right)^{2\rho-1} I_{2\rho-2}^{(n)} \Big\} \quad (\rho \geq 1),$$

$$I_0^{(n)} = \int_{R_1}^{R} r U_0\left(\mu_n \frac{r}{R_1}\right) = N_n.$$

As was shown earlier

$$U_0(k\mu_n) = \frac{2}{\pi k \mathrm{Bi}_2} \cdot \frac{J_0(\mu_n) + \frac{\mu_n}{\mathrm{Bi}_1} J_1(\mu_n)}{J_0(k\mu_n) - \frac{\mu_n}{\mathrm{Bi}_2} J_1(k\mu_n)}.$$

8-4. Method of Calculation of Air Temperature in Cavities of Sections Located Near Shore Contacts of Dam

In § 8-2 and 8-3, we constructed algorithms for determination of the air temperature in the cavities of those sections of a dam located in the middle portion of the structure. It was assumed that the neighboring sections were under identical temperature conditions.

In this section we will study portions of the dam located near the shore contacts. The horizontal cross section of such sections is presented in Figure 8-3. In order to understand the method of solution of the problem, we need only limit ourselves to cavities with flat surroundings.

Let us introduce the subscript ν to represent the number of the cavity and corresponding separating counterforce, if the count is made from the section next to the shore contact of the dam.

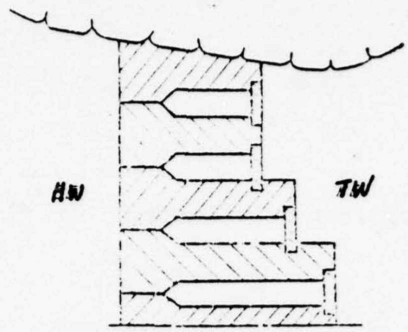

Figure 8-3. Horizontal Section Through Sections with Cavities Located Near Shore Contact of Dam

Composing the thermal balance equation for each of these sections, by the end of the time step $\Delta\tau = \tau_{\ell} - \tau_{\ell-1}$ we arrive at the system of equations

$$a_{\nu,\nu-1}T_{\nu-1} + a_{\nu,\nu}T_{\nu} + a_{\nu,\nu+1}T_{\nu+1} = b_{\nu}\ (\nu = 1, 2, \ldots, s), \tag{8-17}$$

where

$$a_{1,0} = a_{s,s+1} = 0.$$

The solution to this system is the air temperature T_{ν} (ν = 1, 2,..., s) in the cavities of the s sections.

Here

$$a_{\nu,\nu-1} = -\{\lambda SF'(0)\}_{p\nu}\Delta\tau - \{\lambda RSB\}_{p\nu};$$

$$a_{\nu,\nu} = \{\alpha_{n}S_{n}\}_{\nu}\Delta\tau - \{\lambda RSA\}_{0\nu} + \{\lambda SF'(0)\}_{p\nu}\Delta\tau + \tag{8-18}$$

$$+ \{\lambda RS(B - A)\}_{p\nu} + \{\lambda RS(P - W)\}_{p\nu+1} +$$

$$+ \Delta\tau\sum_{j=1}^{m_{\nu}}\{\lambda SF'(0)\}_{j\nu} + \sum_{j=1}^{m_{\nu}}\{\lambda RS(B - A)\}_{j\nu} - \{\lambda SF'(R)\}_{0\nu+1}; \tag{8-19}$$

$$a_{\nu,\nu+1} = \{\lambda SF'(R)\}_{p\nu+1}\Delta\tau + \{\lambda RSW\}_{p\nu+1}; \tag{8-20}$$

$$b_{\nu} = \{q_{1} + q_{2} + \alpha_{n}S_{n}T_{n}\}_{\nu}\Delta\tau + \{s\eta_{r}F'(0)\}_{0\nu}\Delta\tau +$$

$$+ \{SR\eta_{r}B\}_{0\nu} - \{\lambda RSC\}_{0\nu} - \{\lambda SRC\}_{p\nu} + \{\lambda SRV\}_{p\nu+1} +$$

$$+ \Delta\tau\sum_{j=1}^{m_{\nu}}\{\lambda ST_{e}F'(0)\}_{j\nu} - \sum_{j=1}^{m_{\nu}}\{\lambda SRC\}_{j\nu}; \tag{8-21}$$

$$P = \frac{1}{a} \sum_{n=1}^{\infty} \frac{N_n}{\mu_n \| U_0 \|^2} U_1(\mu_n) \left(1 - e^{-\mu_n^2 \frac{a \Delta \tau}{R^2}}\right);$$

$$W = \frac{1}{a} \sum_{n=1}^{\infty} \frac{\bar{F}_{1n}}{\mu_n \| U_0 \|^2} U_1(\mu_n) \left(1 - e^{-\mu_n^2 \frac{a \Delta \tau}{R^2}}\right);$$

$$V = \frac{1}{a} \sum_{n=1}^{\infty} \frac{f_n}{\mu_n \| U_0 \|^2} U_1(\mu_n) \left(1 - e^{-\mu_n^2 \frac{a \Delta \tau}{R^2}}\right);$$

$$U_1(\mu_n) = \frac{\mu_n}{R} N_n - Bi_1.$$

The subscript pν means that the characteristics in the braces relate to the νth separating counterforce; in expressions (8-19), (8-20) and (8-21), summation is conducted with respect to the flat walls ($j = 1, 2, \ldots, m_\nu$) of the νth area ($\nu = 2, 3, \ldots, s-1$), with the exception of the separating counterforces $\{\ \}_{p\nu}$ and the base plate $\{\ \}_{0\nu}$; summation includes in the number m_ν of counterforces $\{\ \}_{p1}$ for the extreme left cavity ($\nu = 1$) and counterforce $\{\ \}_{ps}$ for the extreme right cavity ($\nu = s$), where in the first case T_c is taken equal to the ambient temperature at the counterforce, in the second case -- temperature T_{s+1}, which is the air temperature in the symmetrically located cavity, determined from the solutions of § 8-2.

All remaining symbols are obvious from § 8-2. The basic recommendations of this paragraph also remain unchanged.

The matrix of system (8-17)

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 0 & \cdots 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots a_{s,s-1} & a_{s,s} \end{vmatrix}$$

is a three-diagonal matrix. Its solution is easy to produce by methods of higher algebra [129]. For example, let us take the following approach. We discard the last equation of system (8-17) and find two solutions $T_\nu^{(0)}$ and $T_\nu^{(1)}$ of the remaining truncated system of (s - 1) equations, assuming $T_1^{(0)} = 0$ and $T_1^{(1)} = 1$. This requires that we twice solve a system with a

triangular matrix, not difficult to do.

The expression

$$(T_\nu) = T_\nu^{(0)} + t(T_\nu^{(1)} - T_\nu^{(0)})$$

with any value of parameter t is the solution of the truncated system. We select it so that it satisfies the last equation, discarded earlier

$$a_{s,s-1}T_{s-1} + a_{s,s}T_s = b_s.$$

This gives us

$$t = \frac{b_s - r_s^{(0)}}{r_s^{(1)} - r_s^{(0)}},$$

where

$$r_s^{(0)} = a_{s,s-1}T_{s-1}^{(0)} + a_{s,s}T_s^{(0)};$$

$$r_s^{(1)} = a_{s,s-1}T_{s-1}^{(1)} + a_{s,s}T_s^{(1)}.$$

Consequently, the solution of system (8-17) is

$$T_\nu = T_\nu^{(0)} + \frac{b_s - r_s^{(0)}}{r_s^{(1)} - r_s^{(0)}}(T_\nu^{(1)} - T_\nu^{(0)}),$$

where $T_\nu^{(0)}$ and $T_\nu^{(1)}$ are the solutions of the truncated system of (s - 1) equations, produced on the assumption that $T_1^{(0)} = 0$ and $T_1^{(1)} = 1$; the matrices of these two systems are triangular.

8-5. Method of Calculation of Air Temperature in Cavities with Horizontal Barriers

It is assumed that a section with a cavity is located far from the shore contact of the dam, that the supporting and horizontal walls are flat.

The cross section of such a section is shown in Figure 8-4.

Suppose ν is the number of the cavity (counted from the base), the number of the cover can be seen from Figure 8-4.

Using the same approach as in the previous section, we produce the system of equations

$$a_{\nu,\nu-1}T_{\nu-1}+a_{\nu,\nu}T_{\nu}+a_{\nu,\nu+1}T_{\nu+1}=b_{\nu}\ (\nu=1,2,\ldots,s),$$
$$a_{1,0}=a_{s\,s+1}=0, \tag{8-22}$$

the solution of which is temperature T_ν (ν = 1, 2,..., s) in various zones of the section with horizontal dividers $\{\ \}_{\text{пер}\ \nu+1}$ in the cavity[1].

Here

$$a_{\nu,\nu-1}=-\{\lambda SF'(0)\}_{\text{пер}\,\nu}\Delta\tau-\{\lambda SRB\}_{\text{пер}\,\nu}; \tag{8-23}$$

$$a_{\nu,\nu}=\{\lambda SR(P-W)\}_{\text{пер}\,\nu+1}+\Delta\tau\sum_{j=1}^{m_\nu}\{\lambda SF'(0)\}_{j,\nu}+$$
$$+\sum_{j=1}^{m_\nu}\{\lambda SR(B-A)\}_{j,\nu}+H_\nu; \tag{8-24}$$

$$a_{\nu,\nu+1}=\{\lambda SF'(R)\}_{\text{пер}\,\nu+1}\Delta\tau+\{\lambda SRW\}_{\text{пер}\,\nu+1}; \tag{8-25}$$

$$b_\nu=\{q_1+q_2\}_\nu\Delta\tau+\{\lambda SRV\}_{\text{пер}\,\nu+1}+\Delta\tau\sum_{j=1}^{m_\nu}\{\lambda ST_cF'(0)\}_{j,\nu}-$$
$$-\sum_{j=1}^{m_\nu}\{\lambda SRC\}_{j,\nu}+\Phi_\nu; \tag{8-26}$$

---

[1]The symbol $\{\ \}_{\text{пер}\ \nu+1}$ means that all characteristics contained in the brace relate to the horizontal brace of the νth zone.

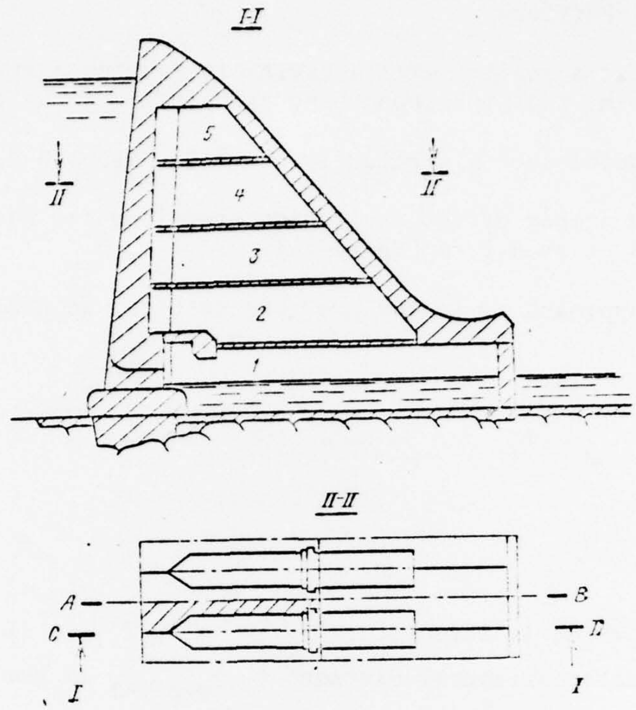

Figure 8-4. Transverse Cross Section of Section with Horizontal Separators

$$H_\nu = \begin{cases} \alpha_в S_в \Delta\tau - \{\lambda SRA\}_0 & \text{where } \nu = 1; \\ \{\lambda SF'(0)\}_{\text{пер}\,\nu} \Delta\tau + \{\lambda SR(B-A)\}_{\text{пер}\,\nu} & \text{where } \nu > 1; \end{cases}$$

$$\Phi_\nu = \begin{cases} \alpha_в S_в T_в \Delta\tau + \{S\eta_1 F'(0)\}_0 \Delta\tau + \{SR\eta_1 B\}_0 - \{\lambda SRC\}_0 & \text{where } \nu = 1; \\ -\{\lambda SRC\}_{\text{пер}\,\nu} & \text{where } \nu > 1. \end{cases}$$

Solution (8-17) is produced by the same method as in § 8-4.

8-6. Certain Additional Problems of the Method of Calculation of the Air Temperature in Cavities in Dams

Possibilities of Simplification of the Calculation Method

In each time step $\Delta\tau = \tau_\ell - \tau_{\ell-1}$, the initial distribution of temperature in elements surrounding the cavity $f(\xi)$ was approximated by a polynomial.

Special calculations have, however, shown that for thin walls, flat and curved, it is sufficient to approximate function $f(\xi)$ by the mean integral temperature through the cross section of the element. The mean integral temperature $\overline{T}_{cp}$ can be calculated from values of temperature at the assigned number of equally spaced points through the cross section of the element at the end of the time step preceding the step in question, i.e., where $\tau = \tau_{\ell-1}$. Then

$$f(\xi) = (\overline{T}_{cp}); \quad \bar{f}_n = (\overline{T}_{cp}) N_n;$$

$$C = \frac{1}{a}\sum_{n=1}^{\infty} \frac{\bar{f}_n}{\mu_n \|U_0\|^2} U_1\left(\mu_n \frac{R_1}{R}\right)\left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}}\right) =$$

$$= (\overline{T}_{cp}) \frac{1}{a}\sum_{n=1}^{\infty} \frac{N_n}{\mu_n \|U_0\|^2} U_1\left(\mu_n \frac{R_1}{R}\right)\left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}}\right) = (\overline{T}_{cp}) A;$$

$$V = \frac{1}{a}\sum_{n=1}^{\infty} \frac{\bar{f}_n}{\mu_n \|U_0\|^2} U_1\left(\mu_n \frac{R_2}{R}\right)\left(1 - e^{-\mu_n^2 \frac{a\Delta\tau}{R^2}}\right) = (\overline{T}_{cp}) P. \qquad (8\text{-}27)$$

Establishment of Quasistable Air Temperature in a Cavity

We present below the results of calculation of the air temperature in the upper portion of a cavity (ignoring the base) in a cross section of a dam such as the Bratsk Power Plant Dam with the following characteristics of elements surrounding the cavity

1 № элемента	2 Размер R_j, м	3 Коэффициент теплоотдачи с наружной средой α_2, ккал/(м²·ч·°C)	4 Примечание
1	13,4	400	5 Контакт с водой
2	7,5	5	6 Ось симметрии
3	2,0	20	7 Контакт с наружным воздухом
4	27,6	20	8 То же

Key: 1, Element No.; 2, Size R_j, m; 3, Heat Transfer Factor with Surrounding Medium α_2, kcal/(m²·hr·C); 4, Note; 5, Interface with Water; 6, Axis of Symmetry; 7, Interface with Air; 8, Same

TABLE 8-1. QUASISTABLE AIR TEMPERATURE IN CAVITY OF DAM

1 Месяц	2 Среднемесячная температура наружного воздуха, °С	3 Время с момента перекрытия, годы: 1		10		25		50		75		100	
		4 Вариант 1	5 Вариант 2	4 Вариант 1	5 Вариант 2	4 Вариант 1	5 Вариант 2	4 Вариант 1	5 Вариант 2	4 Вариант 1	5 Вариант 2	4 Вариант 1	5 Вариант 2
6 Май	7,6	4,135	5,150	-1,369	-1,214	-2,400	-2,390	-2,476	-2,477	-2,478	-2,478	-2,478	-2,478
7 Июнь	15,3	4,588	5,482	0,113	0,260	1,429	-1,485	-1,501	-1,504	-1,505	-1,505	-1,505	-1,505
8 Июль	18,2	5,149	5,987	0,450	0,601	0,551	-0,541	0,625	-0,626	-0,626	0,626	-0,626	-0,626
9 Август	15,3	5,465	6,269	0,978	1,126	0,009	-0,002	-0,082	-0,082	-0,083	-0,083	-0,083	-0,083
10 Сентябрь	7,6	5,359	6,150	1,024	1,170	0,052	0,062	-0,020	-0,020	0,021	-0,021	-0,021	-0,021
11 Октябрь	-3,6	4,769	5,529	0,570	0,711	0,388	-0,378	0,459	0,458	0,159	0,459	-0,459	0,459
12 Ноябрь	-13,6	3,847	4,591	0,246	0,104	-1,190	-1,180	-1,260	-1,260	-1,261	-1,261	1,261	-1,261
13 Декабрь	20,8	2,778	3,507	1,224	-1,684	2,154	-2,145	2,223	-2,224	-2,224	2,224	-2,224	-2,224
14 Январь	23,6	1,869	2,526	2,113	1,975	-3,029	3,019	3,097	-3,096	3,097	3,097	3,097	-3,097
15 Февраль	20,8	1,181	1,885	2,668	2,538	3,571	3,562	3,638	3,638	-3,639	3,639	3,639	3,639
16 Март	13,6	1,041	1,731	2,741	-2,607	3,631	3,622	3,700	3,697	3,697	3,697	3,697	-3,697
17 Апрель	3,6	1,581	2,096	2,335	2,293	-3,211	3,202	3,276	-3,276	3,277	3,277	-3,277	-3,277

Key: 1, Month; 2, Mean Monthly Outside Air Temperature, C; 3, Time from Moment of Coverage, Years; 4, Version 1; 5, Version 2; 6, May; 7, June; 8, July; 9, August; 10, September; 11, October; 12, November; 13, December; 14, January; 15, February; 16, March; 17, April

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We assumed: the heat transfer coefficient on internal surfaces α_1 = 10 kcal/($m^2 \cdot hr \cdot C$); the heat-physical characteristics of the material of the elements: a = 0.003 m^2/hr, λ = 1.7 kcal/($m \cdot hr \cdot C$); the water temperature is constant, 4 C.

It was assumed that the expanded seams of the dam were closed in May. The time step was taken as $\Delta\tau$ = 720 hr.

Two versions were studied: 1) the initial temperature of all elements is constant, 4 C; 2) the initial temperature of the elements is different, 4, 5, 8 and 3 C for the elements as numbered above.

The results of the calculations are presented in Table 8-1. We can see from Table 8-1 that the iterational process suggested for calculation of the quasistable air temperature in the dam does converge.

The quasistable temperature mode is practically set after 25-30 years following closure of the cavity.

Assignment of Thickness of Base Plate

The base model closest to an actual base is a half plane. The use of this model, however, would lead to unjustified complication of algorithms for calculation of the air temperature in a cavity.

In order to simplify calculations, as was noted earlier, the base is replaced with a plate, the thickness of which is assigned from the following condition: further increase in thickness of the plate modeling the base has practically no influence on the calculated value of air temperature in the cavity.

Some results of methodological calculations of air temperature in the cavity of a dam such as that of the Bratsk Power Plant with various values of assigned base plate thickness are presented in Table 8-2.

We can see from Table 8-2 that a change in the base plate thickness from 30 to 60 m has practically very little influence on the calculated air temperature in the cavity.

Comparison of Calculated and Observed Data (On the Example of the Bratsk Power Plant Dam)

Comparison of the results of calculation with the data of field studies was performed on the example of the 30th and 31st sections of the dam of the Bratsk Power Plant. These sections are under observation by a group of workers of the All-Union Scientific Research Institute of Hydrography imeni B. Ye. Vedeneyev, under the leadership of Doctor of Technical Sciences S. Ya. Eydel'man [145]. The air temperature was measured once per month for 1 year

TABLE 8-2. INFLUENCE OF ASSIGNED BASE PLATE THICKNESS ON QUASISTABLE AIR TEMPERATURE, C, IN CAVITY OF DAM

1 Месяц	2 Среднемесячная температура наружного воздуха, °С	3 Назначаемая толщина плиты-основания, м		
		15	30	60
4 Май	7,6	−2,17	−2,18	−2,14
5 Июнь	15,3	−1,24	−1,25	−1,21
6 Июль	18,2	−0,40	−0,41	−0,36
7 Август	15,3	0,11	0,12	0,17
8 Сентябрь	7,6	0,17	0,18	0,24
9 Октябрь	−3,0	−0,24	−0,24	−0,17
10 Ноябрь	−13,0	−1,01	−1,01	−0,94
11 Декабрь	−20,8	−1,93	−1,94	−1,87
12 Январь	−23,6	−2,76	−2,78	−2,72
13 Февраль	−20,8	−3,28	−3,31	−3,25
14 Март	−13,0	−3,33	−3,37	−3,32
15 Апрель	−3,0	−2,93	−2,95	−2,93

Key: 1, Month; 2, Mean Monthly Outside Air Temperature, C; 3, Assigned Base Plate Thickness, m; 4, May; 5, June, 6, July, 7, August, 8, September; 9, October, 10, November; 11, December; 12, January; 13, February; 14, March; 15, April

after closure of the expanded seam. The thermometer was placed at a height of 17 m above the base and a distance of 1 m from the wall of the counterforce. The base was covered with water approximately to the level of the tail water.

The calculations considered the initial temperature of the elements surrounding the cavity, the outside air temperature, water temperature in the head water and tail water and in the base, etc.

TABLE 8-3. COMPARISON OF CALCULATED VALUES OF AIR TEMPERATURE IN EXPANDED SEAM OF BRATSK POWER PLANT DAM WITH FIELD MEASUREMENTS

Месяц	Среднемесячная температура наружного воздуха, °С	Температура в расширенном шве, °С	
		Расчетные значения	Натурные значения
5 Июнь	15,3	4,8	4,7
6 Июль	18,2	5,2	5,0
7 Август	15,3	5,4	[illegible]
8 Сентябрь	7,6	5,2	[illegible]
9 Октябрь	−3,0	4,8	4,7
10 Ноябрь	−13,0	4,2	4,3
11 Декабрь	−20,8	3,6	4,0
12 Январь	−23,6	3,1	3,8
13 Февраль	−20,8	2,8	3,6
14 Март	−13,0	2,0	3,5
15 Апрель	−3,0	3,3	3,5
4 Май	7,6	3,9	4,2

Key: 1, Months; 2, Mean Monthly Air Temperature, C; 3, Temperature in Expanded Seam, C; 3', Calculated; 3", Measured; 4-15, Same as Table 8-2.

As we can see from Table 8-3, the agreement of the calculated values of air temperature in the cavity and the measured values is satisfactory, the divergence not exceeding 0.8 C.

Quasistable Air Temperature in the Cavities of Certain Dams

Table 8-4 presents values of quasistable air temperature in the cavities of various counterforce dams. It was assumed that the dams are under natural conditions, i.e., no special measures are taken to regulate the temperature mode of the structure. For the Kolyma and Zeyskaya Power Plants, the results of calculation studies for various plan versions of the dams are used. Although the Construction of the dam and basic dimensions of the elements are not indicated, the data of Table 8-4 give us a qualitative idea of the air temperature in the cavities of the dams constructed in various regions of the country.

TABLE 8-4. QUASISTABLE AIR TEMPERATURE IN CAVITIES OF CERTAIN DAMS

1 Месяц	2 Братская ГЭС (плотина с расширенными швами)		5 Зейская ГЭС (контрфорсная плотина, проектный вариант)		6 Колымская ГЭС (контрфорсная плотина, проектный вариант)	
	3 Температура наружного воздуха, °C	4 Температура воздуха в полости, °C	3 Температура наружного воздуха, °C	4 Температура воздуха в полости, °C	3 Температура наружного воздуха, °C	4 Температура воздуха в полости, °C
7 Январь	−23,6	−2,78	−31,6	−2,9	−37,9	−9,6
8 Февраль	−20,8	−3,31	−25,1	−3,4	−35,1	−10,0
9 Март	−13,0	−3,37	−15,7	−3,6	−26,4	−10,1
10 Апрель	−3,0	−2,95	−2,0	−3,6	−13,0	−9,8
11 Май	7,6	−2,18	9,1	−3,2	2,0	−9,1
12 Июнь	15,3	−1,25	14,4	−2,7	12,6	−8,2
13 Июль	18,2	−0,41	18,6	−2,2	15,5	−7,4
14 Август	15,3	0,12	16,3	−1,7	12,4	−6,8
15 Сентябрь	7,6	0,18	9,6	−1,4	4,1	−6,7
16 Октябрь	−3,0	−0,24	−2,3	−1,4	−12,1	−7,[illegible]
17 Ноябрь	−13,0	−0,11	−18,4	−1,7	−28,0	−7,9
18 Декабрь	−20,8	−1,94	−26,7	−2,2	−37,1	−8,8

Key: 1, Month; 2, Bratsk Power Plant (Dam with Expanded Seams); 3, Outside Air Temperature, C; 4, Temperature of Air in Cavity, C; 5, Zeyskaya Power Plant Dam (Counterforce Dam, Plan Version); 6, Kolyma Power Plant Dam (Counterforce Dam, Plan Version); 7, January; 8, February; 9, March; 10, April; 11, May; 12, June; 13, July; 14, August; 15, September; 16, October; 17, November; 18, December

CHAPTER 9. SOME PROBLEMS OF CALCULATION OF MOISTURE FIELDS IN CONCRETE DAMS

As we know, the quantity and condition of water in a dam goes far toward determining its physical and mechanical properties: strength, frost resistance, corrosion resistance, modulus of elastic deformation, creep, etc. Water not only participates in the hydration of cement and formation of the micro- and macrostructure of the cement stone as a chemical component, but also interacts with the formed skeleton of the cured concrete.

Changes in the content of water in concrete (due to hydration of the cement, moisture exchange with the environment, etc.) lead to deformations such as shrinkage, frequently causing surface crack formation in concrete structures.

In connection with this, prediction of the moisture mode of structures is of great significance.

We present below methods of calculation of the moisture fields of concrete bodies. We have in mind here only diffusion moisture transfer, i.e., transfer of moisture resulting from the presence of a moisture concentration gradient in the body. The transfer of moisture by filtration, which occurs primarily during the period of use, is not analyzed.

9-1. Calculation of Moisture Fields in Concrete Bodies

The content of moisture in a body is characterized by the moisture function or simply the moisture content $u(x_1, x_2, x_3, \tau)$, by which we mean the mass of moisture in an elementary[1] volume with its center at point (x_1, x_2, x_3) at moment in time τ, related to the mass of absolutely dry matter. The units of measurement of moisture in engineering systems and the SI system are kg/kg.

General Dependences

The basic rule of diffusion moisture transfer in a body is the rule of moisture permeability. For isotropic bodies (and we consider concrete to be such a body) it is formulated as follows: the moisture flux density vector W_m at each point in the field of moisture is proportional to the moisture

[1] Physically rather large.

content gradient at this point

$$W_m = -k_m \gamma_0 \operatorname{grad} u. \tag{9-1}$$

The proportionality factor k_m is called the moisture diffusion coefficient or moisture conductivity coefficient. In formula (9-1), furthermore, γ_0 is the density of the absolutely dry matter.

The equation for unstable moisture conductivity for a heterogeneous isotropic body is:

$$\gamma_0 \frac{\partial u}{\partial \tau} = \operatorname{div}(k_m \gamma_0 \operatorname{grad} u) - w(x_1, x_2, x_3, \tau), \tag{9-2}$$

where w is *the intensity of the internal sinks of moisture,* i.e., the quantities of moisture absorbed by internal sinks per unit volume per unit time.

If the moisture conductivity coefficient k_m and density γ_0 are independent of coordinates and temperature, the moisture conductivity equation is written as:

$$\frac{\partial u}{\partial \tau} = k_m \nabla^2 u - \frac{w}{\gamma_0}. \tag{9-3}$$

The initial moisture content of the body is assigned either as a function of the coordinates

$$u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3), \tag{9-4}$$

or as a constant

$$u(x_1, x_2, x_3, 0) = u_0. \tag{9-4'}$$

The boundary conditions:

of the first kind

$$u\,|_\Gamma = \varphi(\mathscr{P}, \tau); \tag{9-5}$$

of the second kind

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = -\frac{q_m(\mathcal{T},\tau)}{k_m\gamma_0};\tag{9-6}$$

one particular case

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = 0$$ (moisture insulation or symmetry condition);

of the third kind

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = h_m[u_p(\mathcal{T},\tau) - u|_{\Gamma}].\tag{9-7}$$

Here $h_m = \alpha_m/k_m$ is the relative moisture output coefficient; α_m is the moisture output coefficient; u_p is the equillibrium moisture content of the material; n is the external normal at point $\mathcal{T}$ on the surface of the body Γ; ϕ is the moisture content of the surface of the body; η_m is the flux of moisture to the surface of the body.

Boundary conditions of the first, second and third kind can be represented by a single algorithm

$$\alpha\frac{\partial u}{\partial n}\bigg|_{\Gamma} + \beta u|_{\Gamma} = \gamma g(\tau),\tag{9-8}$$

where α, β and γ are constants.

Moisture Physical and Moisture Exchange Characteristics of Concrete

The coefficient of moisture conductivity k_m characterizes the intensity of the process of moisture conduction in a substance and is numerically equal to the density of the flux of moisture where $\gamma_0(\partial u/\partial n) = 1$. The units of measurement of k_m are: engineering system -- m^2/hr, SI -- m^2/s, conversion ratio -- $1\ m^2/hr = 2.7773\cdot10^{-4}\ m^2/s$.

A. V. Belov [10b] for a cement solution with a composition of 1:3 by mass and W/C = 0.45-0.50, produced a value of k_m of $1.31 \cdot 10^{-6}$ to $1.72 \cdot 10^{-6}$ m^2/hr.

S. V. Aleksandrovskiy [2] suggests the following empirical formula for determination of the moisture conductivity coefficient:

$$k_m = 6(1 - 0{,}2W/C)\left(1 + \frac{C - 300}{375}\right) \cdot 10^{-6},\ \text{м}^2/\text{kc}, \quad (9\text{-}9)$$

where W/C is the water/cement ratio; C is the content of cement in the concrete, kg/m^3.

Formula (9-9) was produced by S. V. Aleksandrovskiy as a result of processing his own experimental data. In these experiments, the cement content in the concrete specimens varied from 275 to 375 kg/m^3, W/C from 0.45 to 0.95.

The norms documents [22] recommend for calculation of moisture fields a mean value of moisture conductivity coefficient of $k_m = 5 \cdot 10^{-6}$ m^2/hr.

In the state of equilibrium with moist air, the temperature of a body is equal to the temperature of the air, and its moisture content takes on a certain value u_p, called the equilibrium moisture content.

The equilibrium moisture content of the material u_p is established from the sorption and desorption isotherms[1]. For a number of concretes and materials similar to concrete, the sorption and desorption isotherms have been determined experimentally.

Empirical formulas are also known for calculation of the sorption and desorption isotherms [2, 72]. Thus, for heavy concretes in the area of ordinary above-freezing temperatures, S. V. Aleksandrovskiy [2] recommends the formula

$$u_p = (20 + 1.5\,\phi) \cdot 10^{-4},\ kg/kg,$$

where ϕ is the relative humidity of the air, %.

[1]The desorption isotherm refers to the curve of the equilibrium moisture content of a body as a function of the relative humidity of the air, produced at constant temperature under conditions of moisture efflux; if this dependence is established at constant temperature under conditions of moisture absorption, the curve is called the sorption isotherm.

It is usually assumed that moisture exchange between a body and the environment (air) follows the rule

$$q_m = \alpha_m \gamma_0 (u|_\Gamma - u_p), \tag{9-10}$$

where q_m is the density of the flux of moisture at the surface of the body Γ; α_m is the moisture transfer coefficient.

It follows from (9-10) that the moisture transfer coefficient α_m is numerically equal to the moisture flux density on the surface of the body, related to the product of the density of the absolutely dry matter times the difference between the moisture content of this surface and the equilibrium moisture content. The units of measurement of the moisture transfer coefficient are: engineering system -- m/hr, SI -- m/s, conversion ratio -- 1 m/hr = $2.7773 \cdot 10^{-4}$ m^2/s.

A. V. Belov [10b] for a cement solution under laboratory conditions produced a value of moisture transfer coefficient of $\alpha_m = 2.3 \cdot 10^{-4}$ m/hr.

S. V. Aleksandrovskiy [2] defined the moisture transfer coefficient of a number of concretes as a function of age, water/cement ratio and cement content.

Based on the experiments of S. V. Aleksandrovskiy, the norms documents [22] include the mean value of moisture transfer coefficient for concrete surfaces $\alpha_m = 2 \cdot 10^{-4}$ m/hr.

The influence of the moisture protective properties of a deck, as well as special means of moisture insulation can be considered during calculations of the field of moisture of concrete bodies by introducing the moisture transfer coefficient β_m, defined by the formula

$$\frac{1}{\beta_m} = \frac{1}{\alpha_m} + \sum_{i=1}^{n} \frac{R_{o\pi i}}{k_{mi}}, \tag{9-11}$$

where $R_{o\pi i}$ is the thickness of the ith layer of the deck, the coefficient of moisture conductivity of which is k_{mi}; n is the number of layers in the deck.

Moisture Absorption Intensity Function in Solidifying Concrete

The process of hydration in solidifying concrete is accompanied by chemical bonding of a significant quantity of water, which does not then further participate in moisture transfer. Thus, the process of hydration leads to continuous absorption of moisture throughout the volume of the concrete.

During construction of the moisture absorption function, we will base our calculations on the results of the studies of S. V. Aleksandrovskiy [2], G. D. Vishnevetskiy [20, 21] and I. D. Zaporozhets [50].

G. D. Vishnevetskiy [21], analyzing the data of V. A. Kind, S. D. Okorokov and S. L. Wolfson [56], came to the conclusion that heat liberation in the concrete is proportional to the quantity of chemically bonded water.

S. V. Aleksandrovskiy [2] writes the moisture absorption function as follows:

$$W = KCQ_{sp},\ \text{kg/m}^3,$$

where Q_{sp} is the specific heat liberation function of the cement, kcal/kg; C is the content of cement in the concrete, kg/m^3; K is the proportionality factor, kg/kcal, and for ordinary normal Portland cements we accept a mean value of K of $0.125 \cdot 10^{-2}$ kg/kcal.

I. D. Zaporozhets [50] suggests

$$Q = hB_x,$$

where Q is the heat liberation function in the concrete; B_x is the quantity of chemically bonded water; h is the specific heat liberation of the water as it reacts with the cement.

Processing the data of the experiments of V. A. Kind, S. D. Okorokov and S. L. Wolfson [56], I. D. Zaporozhets calculates the values of h for cements of various mineralogical composition and suggests the formula

$$h = aC_3S + bC_2S + cC_3A + dC_4AF,$$

where C_3S, C_2S, C_3A and C_4AF represent the content of minerals in the clinker, %: a, b, c, d are constants (see § 2-2).

The values of h presented by I. D. Zaporozhets [50] for various cements vary within limits of 300 to 900 kcal/kg, which corresponds to a change in K from $0.1 \cdot 10^{-2}$ to $0.3 \cdot 10^{-2}$ kg/kcal.

Thus, the moisture absorption intensity function in concrete can be represented as

$$w(\tau, T) = pq(\tau, T), \tag{9-12}$$

where p is a certain coefficient, characteristic for concrete of a given composition.

Considering this relationship, based on the results of § 2-2, we can recommend the following expressions for description of the moisture absorption intensity function in the concrete.

a) Moisture absorption depends only on time.

1) Exponential function

$$w = w_0 e^{-m\tau} \ (w_0, m \text{ are parameters});$$

2) Piecewise-exponential function

$$w = w_\nu e^{-m_\nu \tau} \ (\nu = 1, 2, \ldots, i),$$

where w_ν, m_ν are parameters, piecewise-constant functions, defined in $(\tau_{\nu-1}, \tau_\nu)$; i is the number of sectors into which the curve of moisture absorption versus time is divided;

3) The sum of the exponential functions

$$w = \sum_{\nu=1}^{n} w_\nu e^{-\nu m \tau}.$$

b) Moisture absorption depends on time and temperature.

1) The generalized moisture absorption intensity function

$$w = w_\nu (d_\nu + b_\nu T) e^{-m_\nu \tau} \ (\nu = 1, 2, \ldots, i),$$

where w_ν, d_ν, b_ν and m_ν are parameters, piecewise-constant functions, defined in $(\tau_{\nu-1}, \tau_\nu)$;

2) The moisture absorption intensity function after I. D. Zaporozhets

$$w = w_0 2^{\frac{T-20}{10}} \left[1 + A_{20} \int_0^\tau 2^{\frac{T-20}{10}} d\tau\right]^{-1,833},$$

where $w_{20} = 0.833\ pQ_{max}A_{20}$; T is the temperature; p, Q_{max}, A_{20} are parameters.

Statement of the Problem of Moisture Conductivity for Concrete Bodies

The moisture-physical characteristics of concrete in general depend on the moisture content, age and other factors. Under these conditions, the solution of the problem of moisture conductivity by analytic methods is quite difficult. Therefore, so-called zonal methods of calculation are generally used [71]: calculation of unstable moisture conductivity is performed by zones (parts), in each of which the moisture-physical characteristics are assumed constant.

Then the problem of moisture conductivity for concrete bodies is formulated as follows: find the moisture function $u(x_1, x_2, x_3, \tau)$, defined and continuous in the closed area $-R_1 \leq x_1 \leq R_2$, $-L_1 \leq x_2 \leq L_2$, $-D_1 \leq x_3 \leq D_2$, $0 \leq \tau \leq t$, satisfying the equation

$$\frac{\partial u}{\partial \tau} = k_m \nabla^2 u - \frac{1}{\gamma_0} w$$
$$(-R_1 < x_1 < R_2,\ -L_1 < x_2 < L_2,\ -D_1 < x_3 < D_2,\ 0 \leqslant \tau \leqslant t); \quad (9\text{-}13)$$

the initial condition

$$u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3)$$
$$(-R_1 \leqslant x_1 \leqslant R_2,\ -L_1 \leqslant x_2 \leqslant L_2,\ -D_1 \leqslant x_3 \leqslant D_2) \quad (9\text{-}14)$$

and the boundary conditions

$$\alpha \left.\frac{\partial u(x_1, x_2, x_3, \tau)}{\partial n}\right|_\Gamma + \beta u(x_1, x_2, x_3, \tau)|_\Gamma = \gamma g(\circ, \tau). \quad (9\text{-}15)$$

The moisture absorption intensity function in concrete w, contained in equation (9-13), is described by expressions presented above.

As it is easy to see, the edge problem of moisture conductivity is in principle identical to the edge problem of heat conductivity, the solution of which was the subject of Chapters 3 and 4. Analogous and typical calculation plans of structural elements are analyzed in detail in Chapter 4. Therefore, we will not make specific the solution of the problem of moisture conductivity for concrete bodies here. The reader, after assimilating the materials of Chapters 3 and 4, can independently produce these solutions for all of the calculation plans presented in Chapter 4.

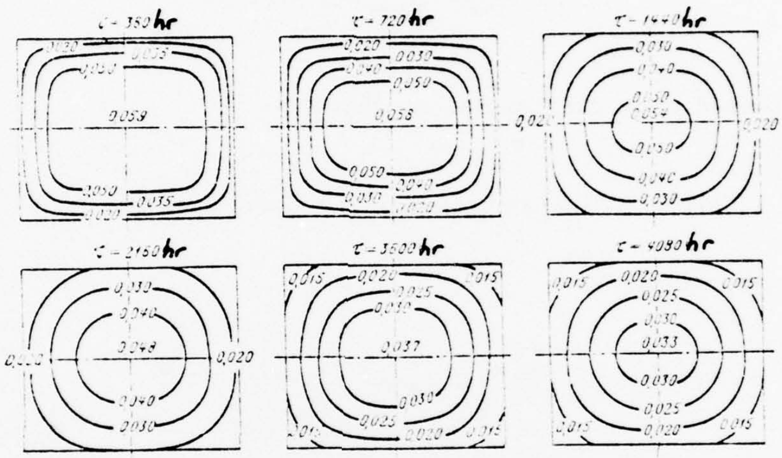

Figure 9-1. Field of Moisture of a Concrete Specimen at Various Moments in Time After Pouring

Figure 9-1 shows the calculated isolines of moisture content over the middle cross section of a laboratory concrete specimen stored in air, calculated using the corresponding analytic solutions. The specimen was poured in a wooden form, then removed after 30 days. Assumptions: the concrete specimen -- cross section 0.5 x 0.3 m, initial moisture content $u_0 = 0.07$ kg/kg, equilibrium moisture content $u_p = 1.25 \cdot 10^{-2}$ kg/kg (corresponding to a relative humidity of the air $\phi = 70\%$), moisture conductivity coefficient of concrete $k_m = 5 \cdot 10^{-6}$ m^2/hr, parameters of intensity function of moisture absorption (assumed exponentially dependent on time) $w_0/\gamma_0 = 0.00115$ 1/hr, m = 0.01 1/hr; form -- thickness 0.04 m, moisture conductivity coefficient $k_{m,2} = 6 \cdot 10^{-2}$ m^2/hr, moisture transfer coefficient $\beta_m = 2 \cdot 10^{-4}$ m/hr.

9-2. Related Heat and Mass Transfer of Concrete Bodies

As many studies have shown, the phenomena of heat transfer and moisture transfer in a capillary porous body such as concrete are mutually related and are described by a system of differential equations.

For the zonal method of calculation of processes of heat and moisture transfer of concrete bodies, this system of equations, after A. V. Lykov [70-72], is

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + \frac{\varepsilon\rho}{c}\frac{\partial u}{\partial \tau} + \frac{1}{c\gamma_0} q(\tau, T);$$
$$\frac{\partial u}{\partial \tau} = k_m \nabla^2 u + k_m \delta \nabla^2 T - \frac{1}{\gamma_0} \omega(\tau, T). \qquad (9\text{-}16)$$

It should be solved with the initial conditions

$$T(x, y, z, 0) = f_1(x, y, z); \; u(x, y, z, 0) = f_2(x, y, z) \qquad (9\text{-}17)$$

and the boundary conditions

$$k_m \gamma_0 \frac{\partial u}{\partial n}\Big|_\Gamma + k_m \gamma_0 \delta \frac{\partial T}{\partial n}\Big|_\Gamma + M\big|_\Gamma = 0;$$
$$-\lambda \frac{\partial T}{\partial n}\Big|_\Gamma + Q\big|_\Gamma - \rho(1 - \varepsilon) M\big|_\Gamma = 0. \qquad (9\text{-}18)$$

In addition to the symbols introduced earlier, here: c is the specific heat capacity of the moist body, calculated per unit mass of absolutely dry bodies; ρ is the specific heat of phase conversion; ε is the criterion of phase conversion; δ is the thermal gradient coefficient; $Q|_\Gamma$ and $M|_\Gamma$ are the specific fluxes of heat and moisture at the surface Γ of the body.

The criterion of phase conversion ε varies from 0 to 1; where $\varepsilon = 0$, we have in mind transfer of moisture only in the form of a liquid, where $\varepsilon = 1$ -- only in the form of water vapor. There is reason to believe that in certain concretes the moisture may move both in the form of vapor (at moisture contents of less than 2%) and in the form of liquid (at moisture contents greater than 4-6%). At moisture contents of 2 to 4%, we must expect movement of the moisture in the form of vapor and in the form of liquid. However, there is some doubt as to the possibility of extending these concepts to heavy, particularly hydraulic engineering, concrete.

The thermal gradient coefficient δ is equal to the ratio of the thermal diffusion coefficient to the moisture conductivity coefficient. Data on the

numerical values of δ for concretes are quite sparse. However, we know that for autoclave concrete (γ = 400 kg/m^3), depending on the moisture content (0.1-0.4 kg/kg), δ varies from $0.40 \cdot 10^{-2}$ to $0.96 \cdot 10^{-2}$ 1/hr, and for foam silicalcite (γ = 900-1000 kg/m^3) it is equal to $0.23 \cdot 10^{-2}$ 1/hr with a moisture content of 0.116 kg/kg and $1.34 \cdot 10^{-2}$ 1/hr with a moisture content of 0.205 kg/kg. In most structural materials, as moisture content decreases, δ drops to 0. However, for example in clays, δ becomes equal to zero at a moisture content of about 2%, and at lower moisture contents changes its sign.

The absence of any reliable data on values of coefficients ε and δ for hydraulic engineering concrete, on the one hand, and the intuitive expectation based on analogies with similar materials -- on the other hand, forces us at this stage of our investigation to assume that for hydraulic engineering concretes, $\varepsilon = 0$ and $\delta = 0$.

Then the system of equations for heat and moisture transfer (9-16) breaks down into individual equations, and the phenomena of heat transfer and mass transfer are unrelated.

The calculations of temperature and moisture fields of concrete masses based on these assumptions ($\varepsilon = 0$ and $\delta = 0$) form the subject of the first eight chapters of this book.

It is not difficult to see that the solutions produced in these chapters can easily be extended to the case when $\delta = 0$, while $\varepsilon \neq 0$.

Where $\delta = 0$ and $\varepsilon \neq 0$, moisture transfer in the body does not depend directly on temperature[1] and the field of moisture content can be established on the basis of the data of § 9-1. As concerns the problem of heat conductivity, consideration of the appearance of certain functions of coordinates in time known from the solution of moisture conductivity problems in the differential equations and in the boundary conditions is quite simple.

However, solution of the full system of related heat and mass transfer equations is of interest, first of all as a theoretical foundation for further and deeper studies of heat and moisture transfer in hydraulic engineering concrete, and secondly, as a more precise form of determination of the temperature and moisture status of structures made of light concrete.

One-Dimensional Problem. Boundary Conditions of First and Second Kind

A study is made of the related heat and moisture exchange of concrete bodies of unlimited wall and unbounded solid type or hollow cylinder type. Due to hydration of the cement, there is heat liberation in the concrete with intensity $q(\tau)$ and water absorption with intensity $w(\tau)$. The initial temperature

[1] If we assume that moisture absorption in the concrete w depends on temperature.

and moisture content are functions of the coordinates. Boundary conditions of the first kind, second kind or "mixed" first and second kind are assigned on the surface of the body.

Establish the temperature and moisture field of the body.

The system of differential equations for heat and moisture transfer

$$\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right)\right] + \frac{\varepsilon\rho}{c}\frac{\partial u}{\partial \tau} + \frac{1}{c\gamma_0}q(\tau);$$
$$\frac{\partial u}{\partial \tau} = k_m\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial u}{\partial \xi}\right)\right] + k_m\delta\left[\frac{1}{\xi^i}\frac{\partial}{\partial \xi}\left(\xi^i\frac{\partial T}{\partial \xi}\right)\right] - \frac{1}{\gamma_0}w(\tau)$$
$$R_1 < \xi < R_2,\ \tau > 0,\ i = 0 \vee 1). \tag{9-19}$$

The initial conditions

$$T(\xi, 0) = f_1(\xi),\ u(\xi, 0) = f_2(\xi),\ (R_1 \leqslant \xi \leqslant R_2). \tag{9-20}$$

The boundary conditions are of one of the following types:

1) First kind

$$T(R_j, \tau) = \varphi_j(\tau);\ u(R_j, \tau) = \psi_j(\tau)\quad (j = 1, 2).$$

2) Second and first kind

$$\frac{\partial T(R_1, \tau)}{\partial \xi} = -\frac{1}{\lambda}\eta_1(\tau),\quad T(R_2, \tau) = \varphi_2(\tau);$$
$$\frac{\partial u(R_1, \tau)}{\partial \xi} + \delta\frac{\partial T(R_1, \tau)}{\partial \xi} = \frac{\eta_{m1}(\tau)}{k_m\gamma_0};\quad u(R_2, \tau) = \psi_2(\tau). \tag{9-21}$$

3) First and second kind

$$T(R_1, \tau) = \varphi_1(\tau);\quad \frac{\partial T(R_2, \tau)}{\partial \xi} = \frac{1}{\lambda}\eta_2(\tau);\quad u(R_1, \tau) = \psi_1(\tau);$$
$$\frac{\partial u(R_2, \tau)}{\partial \xi} + \delta\frac{\partial T(R_2, \tau)}{\partial \xi} = -\frac{\eta_{m2}(\tau)}{k_m\gamma_0}.$$

4) Second kind

$$\frac{\partial T(R_1,\tau)}{\partial \xi} = -\frac{1}{\lambda}\eta_1(\tau); \quad \frac{\partial T(R_2,\tau)}{\partial \xi} = \frac{1}{\lambda}\eta_2(\tau);$$

$$\frac{\partial u(R_1,\tau)}{\partial \xi} + \delta\frac{\partial T(R_1,\tau)}{\partial \xi} = -\frac{\eta_{m1}(\tau)}{k_m\gamma_0}, \quad \frac{\partial u(R_2,\tau)}{\partial \xi} + \delta\frac{\partial T(R_2,\tau)}{\partial \xi} = -\frac{\eta_{m2}(\tau)}{k_m\gamma_0}.$$

Let us first discuss the solution of the problem with the first three types of boundary conditions.

Let us introduce to analysis the substitution function $\Phi(\xi, \tau)$ and assume:

$$\theta(\xi,\tau) = \Phi_1(\xi,\tau) - T(\xi,\tau), \quad \vartheta(\xi,\tau) = \Phi_2(\xi,\tau) - u(\xi,\tau), \tag{9-22}$$

where, depending on the type of boundary conditions

1) Boundary conditions of the first kind

$$\Phi_1(\xi,\tau) = \varphi_1 + (\varphi_2 - \varphi_1)F_a(\xi);$$
$$\Phi_2(\xi,\tau) = \psi_1 + (\psi_2 - \psi_1)F_a(\xi);$$

2) Boundary conditions of the second and first kind

$$\Phi_1(\xi,\tau) = \varphi_2 + \eta_1(\tau)F_b(\xi);$$
$$\Phi_2(\xi,\tau) = \psi_2 - \left[\frac{\eta_{m1}(\tau)}{k_m\gamma_0} + \frac{\delta}{\lambda}\eta_1(\tau)\right]\lambda F_b(\xi);$$

3) Boundary conditions of the first and second kind

$$\Phi_1(\xi,\tau) = \varphi_1 + \eta_2(\tau)F_a(\xi),$$
$$\Phi_2(\xi,\tau) = \psi_1 - \left[\frac{\eta_{m2}(\tau)}{k_m\gamma_0} + \frac{\delta}{\lambda}\eta_2(\tau)\right]\lambda F_a(\xi).$$

The values of the F functions $F(\xi)$ are determined from handbook data as in § 4-2.

We then have a system of differential equations

$$\frac{\partial \theta}{\partial \tau} = a \left[\frac{1}{\xi^\Gamma} \frac{\partial}{\partial \xi} \left(\xi^\Gamma \frac{\partial \theta}{\partial \xi} \right) \right] + \frac{\varepsilon \rho}{c} \frac{\partial \vartheta}{\partial \tau} - Q_1(\xi, \tau);$$

$$\frac{\partial \vartheta}{\partial \tau} = k_m \left[\frac{1}{\xi^\Gamma} \frac{\partial}{\partial \xi} \left(\xi^\Gamma \frac{\partial \vartheta}{\partial \xi} \right) \right] + k_m \delta \left[\frac{1}{\xi^\Gamma} \frac{\partial}{\partial \xi} \left(\xi^\Gamma \frac{\partial \theta}{\partial \xi} \right) \right] + Q_2(\xi, \tau), \tag{9-23}$$

where

$$Q_1(\xi, \tau) = \frac{1}{c\gamma_0} q(\tau) - \Phi'_1(\xi, \tau) + \frac{\varepsilon \rho}{c} \Phi'_2(\xi, \tau);$$

$$Q_2(\xi, \tau) = \frac{1}{\gamma_0} w(\tau) + \Phi'_2(\xi, \tau),$$

with the initial

$$\theta(\xi, 0) = \Phi_1(\xi, 0) - f_1(\xi) = H_1(\xi);$$

$$\vartheta(\xi, 0) = \Phi_2(\xi, 0) - f_2(\xi) = H_2(\xi) \tag{9-24}$$

and boundary conditions

1) of the first kind

$$\theta(R_j, \tau) = \vartheta(R_j, \tau) = 0 \quad (j = 1, 2);$$

2) of the second and first kind

$$\frac{\partial \theta(R_1, \tau)}{\partial \xi} = \frac{\partial \vartheta(R_1, \tau)}{\partial \xi} = 0; \quad \theta(R_2, \tau) = \vartheta(R_2, \tau) = 0; \tag{9-25}$$

3) of the first and second kind

$$\theta(R_1, \tau) = \vartheta(R_1, \tau) = 0; \quad \frac{\partial \theta(R_2, \tau)}{\partial \xi} = \frac{\partial \vartheta(R_2, \tau)}{\partial \xi} = 0.$$

We can apply to problem (9-23)-(9-25) a finite integral transform, defined by the expression

$$\bar{\theta}_n = \int_{R_1}^{R_2} \xi^\Gamma \theta(\xi) U_0 \left(\mu_n \frac{\xi}{R} \right) d\xi, \tag{9-26}$$

where $U_0(\mu_n \frac{\xi}{R})$ is the Eigenfunction of the problem; μ_n is the root of the characteristic equation (see § 3-3).

The system produced after transformation can be reduced to the form

$$\frac{d\bar{\theta}_n}{d\tau} = a_{11}\bar{\theta}_n + a_{12}\bar{\vartheta}_n + \omega_1(\tau);$$
$$\frac{d\bar{\vartheta}_n}{d\tau} = a_{21}\bar{\theta}_n + a_{22}\bar{\vartheta}_n + \omega_2(\tau), \tag{9-27}$$

where

$$a_{11} = \frac{k_m \mu_n^2}{R^2}\left(\mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right);$$
$$a_{12} = -\frac{k_m \mu_n^2}{R^2}\frac{\varepsilon\rho}{c};$$
$$a_{21} = -\frac{k_m \mu_n^2}{R^2}\delta;$$
$$a_{22} = -\frac{k_m \mu_n^2}{R^2};$$
$$\omega_1(\tau) = -\bar{Q}_{1n}(\tau) + \frac{\varepsilon\rho}{c}\bar{Q}_{2n}(\tau);$$
$$\omega_2(\tau) = \bar{Q}_{2n}(\tau);$$
$$\mathrm{Fe} = \frac{\varepsilon\rho\delta}{c}; \quad \mathrm{Lu} = \frac{k_m}{a}.$$

The initial conditions

$$\bar{\theta}_n(0) = \bar{\Phi}_{1n}(\xi, 0) - \bar{f}_{1n} = H_{1n}, \quad \bar{\vartheta}_n(0) = \bar{\Phi}_{2n}(\xi, 0) - f_{2n} = H_{2n}. \tag{9-28}$$

To solve this system of ordinary differential equations (9-27) under the conditions (9-28), we use the method of d'Alambert.

We find

$$\bar{\theta}_n + \gamma_j\bar{\vartheta}_n = \exp[(a_{11} + \gamma_j a_{21})\tau]\{H_{1n} + \gamma_j H_{2n} + \int_0^\tau [\omega_1(\tau) + \gamma_j\omega_2(\tau)]\exp[-(a_{11} + \gamma_j a_{21})\tau]\,d\tau\} \quad (j = 1, 2), \tag{9-29}$$

where γ_j are the roots of the equation

$$a_{12}+\gamma a_{22}=\gamma(a_{11}+\gamma a_{21}). \tag{9-30}$$

It follows from (9-30) that

$$\gamma_j=\frac{(a_{22}-a_{11})+(-1)^j\sqrt{(a_{22}-a_{11})^2+4a_{21}a_{12}}}{2a_{21}}.$$

The discriminant of equation (9-30)

$$D=-4a_{12}a_{21}-(a_{22}-a_{11})^2<0.$$

Consequently, the roots of equation (9-30) are real and different.

Let us tranform the expression beneath the square root

$$(a_{22}-a_{11})^2+4a_{21}a_{12}=(a_{22}+a_{11})^2-4(a_{22}a_{11}-a_{12}a_{21})=$$
$$=\frac{k_m\mu_n^2}{R^2}\left[\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)^2-\frac{4}{\mathrm{Lu}}\right]$$

and represent

$$\nu_j^2=\frac{1}{2}\left\{\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)+(-1)^j\sqrt{\left[\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)^2-\frac{4}{\mathrm{Lu}}\right]}\right\}$$
$$(j=1,\ 2). \tag{9-31}$$

Then, as we can easily see,

$$\gamma_j=\frac{1}{\delta}\left(\nu_j^2-\mathrm{Fe}-\frac{1}{\mathrm{Lu}}\right). \tag{9-32}$$

From this

$$\bar{\vartheta}_n=\frac{\delta}{\nu_2^2-\nu_1^2}\left\{P_{n2}\exp\left[-\mu_n^2\frac{k_m\tau}{R^2}\nu_2^2\right]-P_{n1}\exp\left[-\mu_n^2\frac{k_m\tau}{R^2}\nu_1^2\right]\right\};$$

$$\bar{\theta}_n=-\frac{1}{\nu_2^2-\nu_1^2}\left\{\left(\nu_1^2-\mathrm{Fe}-\frac{1}{\mathrm{Lu}}\right)P_{n2}\exp\left[-\mu_n^2\frac{k_m\tau}{R^2}\nu_2^2\right]-\right.$$
$$\left.-\left(\nu_2^2-\mathrm{Fe}-\frac{1}{\mathrm{Lu}}\right)P_{n1}\exp\left[-\mu_n^2\frac{k_m\tau}{R^2}\nu_1^2\right]\right\}. \tag{9-33}$$

where

$$P_{nj} = H_{1n} + \frac{1}{\delta}\left(\nu_j^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) H_{2n} + \int_0^{\tau} \left\{ \left[\bar{\Phi}'_{1n}(\tau) + \right.\right.$$

$$\left.\left. + \frac{1}{\delta}\left(\nu_j^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) \bar{\Phi}'_{2n}(\tau) \right] \exp\left[\mu_n^2 \frac{k_m \tau}{R^2} \nu_j^2\right] \right\} d\tau -$$

$$- \frac{N_n}{c\gamma_0} \int_0^{\tau} \left\{ \left[q(\tau) - \frac{c}{\delta}\left(\nu_j^2 - \frac{1}{\mathrm{Lu}}\right) \omega(\tau) \right] \exp\left[\mu_n^2 \frac{k_m \tau}{R^2} \nu_j^2\right] \right\} d\tau$$

$$(j = 1,\ 2);$$

$$N_n = \int_{R_1}^{R_2} \xi^{\prime} U_0\left(\mu_n \frac{\xi}{R}\right) d\xi.$$

Keeping in mind the inversion formula

$$v(\xi) = \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} \bar{v}_n U_0\left(\mu_n \frac{\xi}{R}\right) d\xi, \tag{9-34}$$

where $\|U_0\|^2 = \int_{R_1}^{R_2} \xi^{\prime} U_0^2\left(\mu_n \frac{\xi}{R}\right) d\xi$ is the square of the norm of the Eigen-function, and also (9-33), we find the final solution of the problem

$$u = \Phi_2(\xi,\ \tau) - \frac{1}{\nu_2^2 - \nu_1^2} \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \{P_{n2} \exp[-\mu_n^2 \mathrm{Fo}_m \nu_2^2] -$$

$$- P_{n1} \exp[-\mu_n^2 \mathrm{Fo}_m \nu_1^2]\}; \tag{9-35}$$

$$T = \Phi_1(\xi,\ \tau) + \frac{1}{\nu_2^2 - \nu_1^2} \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \left\{\left(\nu_1^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) P_{n2} \times\right.$$

$$\left. \times \exp[-\mu_n^2 \mathrm{Fo}_m \nu_2^2] - \left(\nu_2^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) P_{n1} \right) \exp[-\mu_n^2 \mathrm{Fo}_m \nu^2] \}, \tag{9-36}$$

where

$$\mathrm{Fo}_m = k_m \tau / R^2.$$

Let us now go over to solution of the problem with boundary conditions of the second kind.

In this case

$$\Phi_1(\xi, \tau) = \eta_2(\tau) F_a(\xi) + \eta_1(\tau) F_6(\xi);$$

$$\Phi_2(\xi, \tau) = -\left[\frac{\eta'_2(\tau)}{k_{a}\gamma_0} + \frac{\delta}{\lambda}\eta_2(\tau)\right]\lambda F_a(\xi) -$$

$$-\left[\frac{\eta'_1(\tau)}{k\gamma_0} + \frac{\delta}{\lambda}\eta_1(\tau)\right]\lambda F_6(\xi)\Big]. \tag{9-37}$$

However

$$\nabla^2\Phi_1(\xi, \tau) = \frac{2i}{\lambda(R_2^i + R_1^i)(R_2 - R_1)}\left[R_1^i\,\eta_1(\tau) + R_2^i\,\eta_2(\tau)\right];$$

$$\nabla^2\Phi_2(\xi, \tau) = -\frac{2i}{(R_2^i + R_1^i)(R_2 - R_1)}\left[R_1^i\left(\frac{\eta'_1(\tau)}{k\gamma_0} +\right.\right.$$

$$\left.\left.+\frac{\delta}{\lambda}\eta_1(\tau)\right) + R_2^i\left(\frac{\eta'_2(\tau)}{k\gamma_0} + \frac{\delta}{\lambda}\eta_2(\tau)\right)\right]. \tag{9-38}$$

and therefore to the functions $Q_1(\tau)$ and $Q_2(\tau)$, defined by expressions (9-23), we should add:

$$\frac{2i}{c\gamma_0(R_2^i + R_1^i)(R_2 - R_1)}\left[R_1^i\,\eta_1(\tau) + R_2^i\,\eta_2(\tau)\right]$$

and

$$\frac{2i}{\gamma_0(R_2^i + R_1^i)(R_2 - R_1)}\left[R_1^i\,\eta'_1(\tau) + R_2^i\,\eta'_2(\tau)\right]$$

respectively.

As was noted in § 3-3, with boundary conditions of the second kind the inversion formula is:

$$v(\xi) = \frac{\int_{R_1}^{R_2}\xi^i v(\xi)\,d\xi}{\int_{R_1}^{R_2}\xi^i\,d\xi} + \sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\bar{F}_n U_0\left(\mu_n\frac{\xi}{R}\right)d\xi. \tag{9-39}$$

Consequently, the solution of the problem

$$
\begin{aligned}
u = {} & \Phi_2(\xi,\ \tau) - \frac{1}{\gamma_0}\int_0^{\tau} w(\tau)\,d\tau - \frac{2^{t}}{(R_2^{t} + R_1^{t})\,(R_2 - R_1)} \times \\
& \times \left[\int_{R_1}^{R_2} \xi^{t} H_2(\xi)\,d\xi + \frac{1}{\gamma_0}\int_0^{\tau} (R_1^{t}\,\eta_{m1}(\tau) + R_2^{t}\,\eta_{m2}(\tau))\,d\tau + \right. \\
& \left. + \int_0^{\tau}\int_{R_1}^{R_2} \xi^{t}\Phi'_2(\xi,\ \tau)\,d\tau\,d\xi \right] - \frac{\delta}{\nu_2^2 - \nu_1^2}\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\,U_0\left(\mu_n\frac{\xi}{R}\right) \times \\
& \times \{P_{n2}\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_2^2] - P_{n1}\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_1^2]\};
\end{aligned}
\tag{9-40}
$$

$$
\begin{aligned}
T = {} & \Phi_1(\xi,\ \tau) + \frac{1}{c\gamma_0}\int_0^{\tau} [q(\tau) - \varepsilon\rho w(\tau)]\,d\tau - \\
& - \frac{2^{t}}{(R_2^{t} + R_1^{t})\,(R_2 - R_1)} \left[\int_{R_1}^{R_2} \xi^{t} H_1(\xi)\,d\xi - \frac{\varepsilon\rho}{c\gamma_0}\int_0^{\tau} (R_1^{t}\,\eta_1(\tau) + \right. \\
& \left. + R_2^{t}\,\eta_2(\tau))\,d\tau + \int_0^{\tau}\int_{R_1}^{R_2} \xi^{t}\Phi'_1(\xi,\ \tau)\,d\tau\,d\xi \right] + \\
& + \frac{1}{\nu_2^2 - \nu_1^2}\sum_{n=1}^{\infty}\frac{1}{\|U_0\|^2}\,U_0\left(\mu_n\frac{\xi}{R}\right)\left\{ \left(\nu_1^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) \times \right. \\
& \left. \times P_{n2}\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_2^2] - \left(\nu_2^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) P_{n1}\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_1^2] \right\},
\end{aligned}
\tag{9-41}
$$

where

$$
\begin{aligned}
P_{nj} = {} & H_1 + \frac{1}{\delta}\left(\nu_j^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right) H_{2n} + \int_0^{\tau} \left\{ \overline{\Phi}'_{1n}(\tau) + \right. \\
& \left. + \frac{1}{\delta}\left(\nu_j^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right)\overline{\Phi}'_{2n}(\tau) \right\} \exp\left[\mu_n^2\frac{k_m\tau}{R^2}\nu_j^2\right] d\tau - \\
& - \frac{N_n}{c\gamma_0}\int_0^{\tau} \left\{ \left[q(\tau) - \frac{c}{\delta}\left(\nu_j^2 - \frac{1}{\mathrm{Lu}}\right) w(\tau) \right] + \right. \\
& + \frac{2^{t}}{(R_2^{t} + R_1^{t})\,(R_2 - R_1)} \left[(R_1^{t}\,\eta_1(\tau) + R_2^{t}\,\eta_2(\tau)) - \right. \\
& \left.\left. - \frac{c}{\delta}\left(\nu_j^2 - \frac{1}{\mathrm{Lu}}\right)(R_1^{t}\,\eta_{m1}(\tau) + R_2^{t}\,\eta_{m2}(\tau)) \right] \right\} \times \\
& \times \exp\left[\mu_n^2\frac{k_m\tau}{R^2}\nu_j^2\right] d\tau \quad (j = 1,2).
\end{aligned}
$$

In conclusion, as an example let us present the final solution of the problem of the related heat and mass transfer with boundary conditions of the second kind for a hollow cylinder produced in [163] under the following conditions: $T(r, 0) = f_1(r) = T_0 = \mathrm{const}$; $u(r, 0) = f_2(r) = u_0 = \mathrm{const}$; $q(\tau) = w(\tau) = 0$;

$$\eta_1(\tau) = \eta_{01} = \mathrm{const};\ \eta_2(\tau) = \eta_{02} = \mathrm{const};$$

$$\eta_{m1}(\tau) = \eta_{m01} = \mathrm{const};\ \eta_{m2} = \eta_{m02} = \mathrm{const}.$$

This solution is as follows:

$$\frac{u_0 - u}{u_0} = \frac{2(k\eta_{m021} + 1)}{k^2 - 1}\,\mathrm{Ki}_m\mathrm{Fo}_m + \frac{\mathrm{Ki\,Pn}}{\mathrm{Lu}}\,\frac{Q_{021}(r)}{\nu_2^2\nu_1^2(k^2 - 1)} +$$

$$+ \frac{\mathrm{Ki}_m}{\mathrm{Lu}}\,\frac{M_{021}(r)}{\nu_2^2\nu_1^2(k^2 - 1)} - \frac{\mathrm{Ki\,Pn}}{\mathrm{Lu}}\,\frac{\pi}{\nu_2^2 - \nu_1^2}\sum_{n=1}^{\infty}\{\mu_n[J_1^2(\mu_n) - J_1^2(k\mu_n)]\}^{-1}\times$$

$$\times[\eta_{021}J_1(\mu_n) + J_1(k\mu_n)]\,J_1(k\mu_n)\,U_0\left(\mu_n\frac{r}{R_1}\right)\times$$

$$\times\left\{\frac{1}{\nu_2^2}\exp[-\mu_n^2\mathrm{Fo}_m\nu_2^2] - \frac{1}{\nu_1^2}\exp[-\mu_n^2\mathrm{Fo}_m\nu_1^2]\right\} +$$

$$+ \mathrm{Ki}_m\frac{\pi}{\nu_2^2 - \nu_1^2}\sum_{n=1}^{\infty}\{\mu_n[J_1^2(\mu_n) - J_1^2(k\mu_n)]\}^{-1}\times$$

$$\times[\eta_{m021}J_1(\mu_n) + J_1(k\mu_n)]\,J_1(k\mu_n)\,U_0\left(\mu_n\frac{r}{R_1}\right)\times$$

$$\times\left\{\left(1 - \frac{1}{\nu_2^2\mathrm{Lu}}\right)\exp[-\mu_n^2\mathrm{Fo}_m\nu_2^2] - \left(1 - \frac{1}{\nu_1^2\mathrm{Lu}}\right)\times\right.$$

$$\left.\times\exp[-\mu_n^2\mathrm{Fo}_m\nu_1^2]\right\}; \tag{9-42}$$

$$\frac{T - T_0}{T_0} = \frac{2(k\eta_{021} + 1)}{k^2 - 1}\,\mathrm{KiFo} - \frac{2(k\eta_{m021} + 1)}{k^2 - 1}\,\varepsilon\mathrm{Ki}_m\mathrm{Ko}_m\mathrm{Fo}_m +$$

$$+ \frac{\mathrm{Ki}}{\mathrm{Lu}}\,\frac{Q_{021}(r)}{\nu_2^2\nu_1^2(k^2 - 1)} + \frac{\mathrm{Ki}}{\mathrm{Lu\,Pn}}\,\frac{\varepsilon M_{021}(r)}{\nu_2^2\nu_1^2(k^2 - 1)} +$$

$$+ \frac{\mathrm{Ki}}{\mathrm{Lu}}\,\frac{\pi}{\nu_2^2 - \nu_1^2}\sum_{n=1}^{\infty}\{\mu_n[J_1^2(\mu_n) - J_1^2(k\mu_n)]\}^{-1}[\eta_{m021}J_1(\mu_n) +$$

$$+ J_1(k\mu_n)]\,J_1(k\mu_n)\,U_0\left(\mu_n\frac{r}{R_1}\right)\left\{\frac{1}{\nu_2^2}\exp[-\mu_n^2\mathrm{Fo}_m\nu_2^2] -\right.$$

$$\left.- \frac{1}{\nu_1^2}\exp[-\mu_n^2\mathrm{Fo}_m\nu_1^2]\right\} +$$

$$
+\frac{\mathrm{Ki}_m}{\mathrm{Pn}}\,\frac{\pi}{\nu_2^2-\nu_1^2}\sum_{n=1}^{\infty}\{\mu_n\,[J_1^2(\mu_n)-J_1^2(k\mu_n)]\}^{-1}\times
$$

$$
\times\,[\eta_{m\,21}J_1(\mu_n)+J_1(k\mu_n)]\,J_1(k\mu_n)\,U_0\left(\mu_n\frac{r}{R_1}\right)\times
$$

$$
\times\left\{\left(1-\frac{1}{\nu_2^2\,\mathrm{Lu}}\right)\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_2^2]-\right.
$$

$$
\left.-\left(1-\frac{1}{\nu_1^2\,\mathrm{Lu}}\right)\exp[-\mu_n^2\,\mathrm{Fo}_m\,\nu_1^2]\right\}, \tag{9-43}
$$

where

$$
k=\frac{R_2}{R_1};\quad \eta_{m021}=\frac{\eta_{m02}}{\eta_{m01}};\quad \eta_{021}=\frac{\eta_{02}}{\eta_{01}};\quad \mathrm{Fo}_m=\frac{k_m\tau}{R_1^2};\quad \mathrm{Fo}=\frac{a\tau}{R_1^2};
$$

$$
\mathrm{Ki}=\frac{R_1\eta_{01}}{\lambda T_0},\quad \mathrm{Ki}_m=\frac{R_1\eta_{m01}}{k_m\gamma_0 u_0};\quad \mathrm{Lu}=\frac{k_m}{a};\quad \mathrm{Ko}=\frac{\rho u_0}{cT_0};\quad \mathrm{Pn}=\frac{\delta T_0}{u_0};
$$

μ_n is the root of the characteristic equation

$$
Y_1(\mu_n)\,J_1(k\mu_n)-J_1(\mu_n)\,Y_1(k\mu_n)=0;
$$

$$
U_0\left(\mu_n\frac{r}{R_1}\right)=Y_1(\mu_n)\,J_0\left(\mu_n\frac{r}{R_1}\right)-J_1(\mu_n)\,Y_0\left(\mu_n\frac{r}{R_1}\right);
$$

$$
Q_{021}(r)=\frac{k\eta_{021}+1}{2}\left(\frac{r^2}{R_1^2}-\frac{k^2+1}{2}\right)+
$$

$$
+k(\eta_{021}+k)\left(\frac{k^2\ln k}{k^2-1}-\ln\frac{r}{R_1}-\frac{1}{2}\right);
$$

$$
M_{021}(r)=\frac{k\eta_{m021}+1}{2}\left(\frac{r^2}{R_1^2}-\frac{k^2+1}{2}\right)+
$$

$$
+k(\eta_{m021}+k)\left(\frac{k^2\ln k}{k^2-1}-\ln\frac{r}{R_1}-\frac{1}{2}\right).
$$

One-Dimensional Problem. Boundary Conditions of the Third Kind

For simplicity, let us limit ourselves to analysis of the heat and moisture state of an unlimited symmetrical wall with constant initial and boundary conditions.

The system of differential equations

$$\frac{\partial T}{\partial \tau}=a\,\frac{\partial^2 T}{\partial x^2}+\frac{\varepsilon\rho}{c}\,\frac{\partial u}{\partial \tau}+\frac{1}{c\gamma_0}\,q_0e^{-m\tau};$$

$$\frac{\partial u}{\partial \tau}=k_m\,\frac{\partial^2 u}{\partial x^2}+k_m\delta\,\frac{\partial^2 T}{\partial x^2}-\frac{1}{\gamma_0}\,w_0e^{-m\tau}. \tag{9-44}$$

The initial conditions

$$T(x,\,0)=T_0;\ u(x,\,0)=u_0. \tag{9-45}$$

The boundary conditions

$$-\lambda\frac{\partial T(R,\,\tau)}{\partial x}+\alpha\,[T_c-T(R,\,\tau)]+(1-\varepsilon)\,\rho\alpha_m\gamma_0\,[u_p-u(R,\,\tau)]=0;$$

$$k_m\,\frac{\partial u(R,\,\tau)}{\partial x}+k_m\delta\,\frac{\partial T(R,\,\tau)}{\partial x}-\alpha_m\,[u_p-u(R,\,\tau)]=0; \tag{9-46}$$

$$\frac{\partial T(0,\,\tau)}{\partial x}=\frac{\partial u(0,\,\tau)}{\partial x}=0.$$

We assume

$$T=T_1+T_2;\ u=u_1+u_2,$$

where T_1 and u_1 satisfy the full system of equations (9-44) with the zero initial

$$T_1(x,\,0)=0,\ u_1(x,\,0)=0$$

and boundary

$$\lambda\frac{\partial T_1(R,\,\tau)}{\partial x}+\alpha T_2(R,\,\tau)+(1-\varepsilon)\,\rho\alpha_m\gamma_0u_1(R,\,\tau)=0;$$

$$k_m\,\frac{\partial u_1(R,\,\tau)}{\partial x}+k_m\delta\,\frac{\partial T_2(R,\,\tau)}{\partial x}+\alpha_m u_1(R,\,\tau)=0;$$

$$\frac{\partial T_1(0,\,x)}{\partial x}=\frac{\partial u_1(0,\,x)}{\partial x}=0$$

conditions, while T_2 and u_2 satisfy equation system (9-44) without free terms and heterogeneous edge conditions of type (9-45) and (9-46).

The solution for the functions T_1 and u_1 is produced using the Laplace transform

$$\bar{v} = \int_0^\infty v e^{-s\tau} d\tau.$$

We have:

$$s\bar{T}_1 = a\bar{T}_1'' + \frac{\varepsilon\rho}{c} s\bar{u} + \frac{1}{s+m}\frac{q_0}{c\gamma_0};$$

$$s\bar{u}_1 = k_m \bar{u}''_1 + k_m \delta \bar{T}_1'' - \frac{1}{s+m}\frac{w_0}{\gamma_0};$$

$$\lambda \bar{T}'_1(R, s) + \alpha \bar{T}_1(R, s) + (1-\varepsilon)\rho\alpha_m \gamma_0 \bar{u}_1(R, s) = 0;$$

$$k_m \bar{u}'_1(R, s) + k_m \delta \bar{T}_1'(R, s) + \alpha_m \bar{u}_1(R, s) = 0;$$

$$\bar{T}_1'(0, s) = \bar{u}'_1(0, s) = 0.$$

From this

$$\bar{T}_1 - \frac{1}{s(s+m)}\frac{1}{c\gamma_0}(q_0 - w_0\varepsilon\rho) = B_1 \operatorname{ch}\sqrt{\frac{s}{a}}\nu_1 x + B_2 \operatorname{ch}\sqrt{\frac{s}{a}}\nu_2 x;$$

$$\bar{u}_1 + \frac{1}{s(s+m)}\frac{1}{\gamma_0} w_0 = \frac{c}{\varepsilon\rho}(1-\nu_1^2) B_1 \operatorname{ch}\sqrt{\frac{s}{a}}\nu_1 x + \frac{c}{\varepsilon\rho}(1-\nu_2^2) B_2 \operatorname{ch}\sqrt{\frac{s}{a}}\nu_2 x,$$

where

$$B_j = (-1)^{j\pm 1}\{c\gamma_0 s(s+m)[Q_2^{(h)}P_1^{(h)} - Q_1^{(h)}P_2^{(h)}]\}^{-1} \times \{q_0 P_{j\pm 1}^{(h)} + w_0\varepsilon\rho[Q_{j\pm 1}^{(h)} - (1-K_1)P_{j\pm 1}^{(h)}]\};$$

$$P_j^{(h)} = (1-\nu_j^2)\operatorname{ch}\sqrt{\frac{s}{a}}\nu_j R + \frac{1}{\mathrm{Bi}_m}[(1-\nu_j^2)+\mathrm{Fe}] \times \sqrt{\frac{s}{a}}\nu_j R \operatorname{sh}\sqrt{\frac{s}{a}}\nu_j R;$$

$$Q_j^{(h)} = [1 + (1-\nu_j^2)K_1]\operatorname{ch}\sqrt{\frac{s}{a}}\nu_j R + \frac{1}{\mathrm{Bi}}\sqrt{\frac{s}{a}}\nu_j R \operatorname{sh}\sqrt{\frac{s}{a}}\nu_j R; \quad (9\text{-}47)$$

$$K_1 = \frac{1-\varepsilon}{\varepsilon}\mathrm{Lu}\frac{\mathrm{Bi}_m}{\mathrm{Bi}}; \quad \mathrm{Bi} = \frac{\alpha R}{\lambda}; \quad \mathrm{Bi}_m = \frac{\alpha_m R}{k_m};$$

ν_j is defined by expression (9-31).

Note. In formula (9-47), where j = 1 we should assume j ± 1 = 2, where j = 2, j ± 1 = 1.

The originals of T_1 and u_1 are found using the second theorem of expansion (see § 3-4).

The result

$$T_1 = -\frac{R^2}{\lambda m^{*2}}\left[(q_0 - \varepsilon\rho w_0) + \frac{\mathcal{F}_2(m^*)\cos\nu_1 m^*\frac{x}{R} - \mathcal{F}_1(m^*)\cos\nu_2 m^*\frac{x}{R}}{\Delta(m^*)}\right]\times$$
$$\times e^{-m^{*2}\mathrm{Fo}} + \frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{1}{m^{*2}-\mu_n^2}\left[C_{n2}\cos\nu_1\mu_n\frac{x}{R} - \right.$$
$$\left. - C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right] e^{-\mu_n^2\mathrm{Fo}}; \tag{9-48}$$

$$u_1 = \frac{R^2}{\lambda m^{*2}}\,\frac{c}{\varepsilon\rho}\left[\varepsilon\rho w_0 - \right.$$
$$\left. - \frac{(1-\nu_1^2)\mathcal{F}_2(m^*)\cos\nu_1 m^*\frac{x}{R} - (1-\nu_2^2)\mathcal{F}_1(m^*)\cos\nu_2 m^*\frac{x}{R}}{\Delta(m^*)}\right]\times$$
$$\times e^{-m^{*2}\mathrm{Fo}} + \frac{c}{\varepsilon\rho}\,\frac{R^2}{\lambda}\sum_{n=1}^{\infty}\frac{1}{m^{*2}-\mu_n^2}\left[C_{n2}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R} - \right.$$
$$\left. - C_{n1}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}\right] e^{-\mu_n^2\mathrm{Fo}}, \tag{9-49}$$

$$C_{nj} = \frac{2\mathcal{F}_j(\mu_n)}{\mu_n\Psi_n}\ (j = 1,2); \tag{9-50}$$

$$\mathcal{F}_j(l) = q_0 P_j(l) + w_0\varepsilon\rho[Q_j(l) - (1+K_1)P_j(l)]\ (l = m^*,\ \mu_n); \tag{9-51}$$

$$Q_j(l) = [1 + (1-\nu_j^2)K_1]\cos\nu_j l - \frac{1}{\mathrm{Bi}}\nu_j l\sin\nu_j l; \tag{9-52}$$

$$P_j(l) = (1-\nu_j^2)\cos\nu_j l - \frac{1}{\mathrm{Bi}_m}[(1-\nu_j^2) + \mathrm{Fe}]\nu_j l\sin\nu_j l; \tag{9-53}$$

$$\Psi_n = \nu_1 A_1 P_2(\mu_n) + \nu_2 B_2 Q_1(\mu_n) - \nu_2 A_2 P_1(\mu_n) - \nu_1 B_1 Q_2(\mu_n); \tag{9-54}$$

$$A_j = \left[1 + \frac{1}{\mathrm{Bi}} + (1-\nu_j^2)K_1\right]\sin\nu_j\mu_n + \frac{\mu_n\nu_j}{\mathrm{Bi}}\cos\nu_j\mu_n; \tag{9-55}$$

$$B_j = (1 - \nu_j^2)\sin \nu_j\mu_n + \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_{an}}(\sin \nu_j\mu_n + \nu_j\mu_n \cos \nu_j\mu_n);$$

$$\Delta(m^*) = \begin{vmatrix} P_1(m^*) & P_2(m^*) \\ Q_1(m^*) & Q_2(m^*) \end{vmatrix}; \quad (9\text{-}56)$$

(9-57)

$$m^* = \frac{mR^2}{a}; \quad \mathrm{Fo} = \frac{a\tau}{R^2};$$

μ_n is the root of the characteristic equation in the form $\Delta(\mu_n) = 0$

or

$$\frac{M}{N} = \frac{\mu_n}{\mathrm{Bi}_m}, \quad (9\text{-}58)$$

where

$$M = P_1(\mu_n)\cos \nu_2\mu_n - P_2(\mu_n)\cos \nu_1\mu_n; \quad (9\text{-}59)$$

$$N = \left[(1 - \nu_1^2)\frac{1-\varepsilon}{\varepsilon}\mathrm{Lu}\,\mathrm{Bi}_m \frac{\cos \nu_1\mu_n}{\mu_n} - \nu_1 \sin \nu_1\mu_n\right] P_2(\mu_n) -$$

$$-\left[(1 - \nu_2^2)\frac{1-\varepsilon}{\varepsilon}\mathrm{Lu}\,\mathrm{Bi}_m \frac{\cos \nu_2\mu_n}{\mu_n} - \nu_2 \sin \nu_2\mu_n\right] P_1(\mu_n). \quad (9\text{-}60)$$

The solution of the problem of functions T_2 and u_2 is presented in the monograph of A. V. Lykov and Yu. A. Mikhaylov [71] and has the form

$$\frac{T_2(x, \tau) - T_0}{T_c - T_0} = 1 - \sum_{n=1}^{\infty}\left[D_{n2}\cos \nu_1\mu_n \frac{x}{R} - D_{n1}\cos \nu_2\mu_n \frac{x}{R}\right] e^{-\mu_n^2 \mathrm{Fo}}; \quad (9\text{-}61)$$

$$\frac{u_0 - u_2(x, \tau)}{u_0 - u_p} = 1 + \frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{n=1}^{\infty}\left[D_{n2}(1 - \nu_1^2)\cos \nu_1\mu_n \frac{x}{R} - \right.$$

$$\left. - D_{n2}(1 - \nu_2^2)\cos \nu_2\mu_n \frac{x}{R}\right] e^{-\mu_n^2 \mathrm{Fo}}, \quad (9\text{-}62)$$

where

$$D_{nj} = \frac{2}{\mu_n \Psi_n}[(1 - \varepsilon\,\mathrm{Ko}\,K_1)P_j(\mu_n) - \varepsilon\,\mathrm{Ko}\,Q_j(\mu_n)] \quad (j = 1,\ 2); \quad (9\text{-}63)$$

$$\mathrm{Ko} = \frac{\delta u_0}{cT_0}; \quad \mathrm{Fo} = \frac{a\tau}{R^2};$$

the remaining symbols are the same as before.

Two-Dimensional Problem

The method of solution will be illustrated on the example of the problem for a rectangle with boundary conditions of the second and first kind.

The system of equations of related heat and mass transfer

$$\frac{\partial T}{\partial \tau}=a\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial T}{\partial \xi}\right)+\frac{\partial^2 T}{\partial y^2}\right]+\frac{\varepsilon\rho}{c}\frac{\partial u}{\partial \tau}+\frac{1}{c\gamma_0}q(\tau);$$

$$\frac{\partial u}{\partial \tau}=k_{m}\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial u}{\partial \xi}\right)+\frac{\partial^2 u}{\partial y^2}\right]+k_{m}\delta\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial T}{\partial \xi}\right)+\right.$$
$$\left.+\frac{\partial^2 T}{\partial y^2}\right]-\frac{1}{\gamma_0}w(\tau)\quad (R_1<\xi<R_2,\ 0<y<L,\ \tau>0;\ i=0 \vee 1). \tag{9-64}$$

The initial conditions

$$T(\xi,\ y,\ 0)=f_1(\xi,\ y);\ u(\xi,\ y,\ 0)=f_2(\xi,\ y)\ (R_1\leq\xi\leq R_2,\ 0\leq y\leq L). \tag{9-65}$$

The boundary conditions

$$\frac{\partial T(R_1,\ y,\ \tau)}{\partial \xi}=-\frac{1}{\lambda}\eta_1(\tau);\quad \frac{\partial u(R_1,\ y,\ \tau)}{\partial \xi}+\delta\frac{\partial T(R_1,\ y,\ \tau)}{\partial \xi}=\frac{\eta_{m1}(\tau)}{k_{m}\gamma_0};$$
$$T(R_2,\ y,\ \tau)=\varphi_2(\tau);\ u(R_2,\ y,\ \tau)=\psi_2(\tau);$$
$$T(\xi,\ 0,\ \tau)=\varphi_3(\tau);\ u(\xi,\ 0,\ \tau)=\psi_3(\tau);$$
$$T(\xi,\ L,\ \tau)=\varphi_4(\tau);\ u(\xi,\ L,\ \tau)=\psi_4(\tau). \tag{9-66}$$

We assume

$$\theta_1=\Phi_1(\xi,\ \tau)-T;\ \theta_2=\Phi_2(\xi,\ \tau)-u, \tag{9-67}$$

where

$$\Phi_1(\xi,\ \tau)=\varphi_2(\tau)+\eta_1(\tau)F_{b}(\xi);$$
$$\Phi_2(\xi,\ \tau)=\psi_2(\tau)-\left[\frac{\eta_{m1}(\tau)}{k_{m}\gamma_0}+\frac{\delta}{\lambda}\eta_1(\tau)\right]\lambda F_{b}(\xi); \tag{9-68}$$

the F function $F_b(\xi)$ is determined by the data of § 4-2.

Using a finite integral transform such as (9-26), we produce a system of differential equations

$$\frac{\partial \bar\theta_{1n}}{\partial \tau} = a\,\frac{\partial^2 \bar\theta_{1n}}{\partial y^2} - \frac{a\mu_n^2}{R^2}\,\bar\theta_{1n} + \frac{\varepsilon\rho}{c}\,\frac{\partial \bar\vartheta_{1n}}{\partial \tau} - \bar Q_{1n}(\tau);$$

$$\frac{\partial \bar\vartheta_{1n}}{\partial \tau} = k_{m}\,\frac{\partial^2 \bar\vartheta_{1n}}{\partial y^2} + k_{m}\delta\,\frac{\partial^2 \bar\theta_{1n}}{\partial y^2} - \frac{k_{m}\mu_n^2}{R^2}\,\bar\vartheta_{1n} - \frac{k_{m}\delta\mu_n^2}{R^2}\,\bar\theta_{1n} + \bar Q_{2n}(\tau),$$

where

$$\bar Q_{1n}(\tau) = \left(\frac{1}{c\gamma_0}\,q + \frac{\varepsilon\rho}{c}\,\psi_2 - \varphi'_2\right)N_n - \left(\frac{\eta'_1}{\lambda} + \mathrm{Fe}\,\frac{\eta_1}{\lambda} + \right.$$
$$\left. + \frac{\varepsilon\rho}{k_{m}c\gamma_0}\,\eta_{m1}\right)\lambda \bar F_{\sigma},$$
$$\bar Q_{2n}(\tau) = \left(\frac{1}{\gamma_0}\,w + \psi'_2\right)N_n - \left(\frac{\delta}{\lambda}\,\eta_1 + \frac{\eta_{m1}}{k_{m}\gamma_0}\right)\lambda \bar F_b$$

with the initial

$$\bar\theta_{1n}(y,\,0) = \varphi_2(0)\,N_n + \eta_1(0)\,F_b - \bar f_{1n} = H_{1n};$$
$$\bar\vartheta_{1n}(y,\,0) = \psi_2(0)\,N_n - \left[\frac{\eta_{m1}(0)}{k_{m}\gamma_0} + \frac{\delta}{\lambda}\,\eta_1(0)\right]\lambda F_b -$$
$$- \bar f_{2n} = H'_{2n}$$

and boundary conditions

$$\bar\theta_{1n}(0,\,\tau) = (\varphi_2 - \varphi_3)\,N_n + \eta_1 F_b;$$
$$\bar\vartheta_{1n}(0,\,\tau) = (\psi_2 - \psi_3)\,N_n - \left[\frac{\eta_{m1}}{k_{m}\gamma_0} + \frac{\delta}{\lambda}\,\eta_1\right]\lambda F_b,$$
$$\bar\theta_{1n}(L,\,\tau) = (\varphi_2 - \varphi_4)\,N_n + \eta_1 F_b,$$
$$\bar\vartheta_{1n}(L,\,\tau) = (\psi_2 - \psi_4)\,N_n - \left[\frac{\eta_{m1}}{k_{m}\gamma_0} + \frac{\delta}{\lambda}\,\eta_1\right]\lambda F_b.$$

The substitution

$$\bar\theta_{2n}(y,\,\tau) = \Phi_3(y,\,\tau) - \bar\theta_{1n},\quad \bar\vartheta_{2n}(y,\,\tau) = \Phi_4(y,\,\tau) - \bar\vartheta_{1n},$$

where

$$\Phi_3(y, \tau) = (\varphi_2 - \varphi_3) N_n + \eta_1 \bar{F}_\delta + N_n (\varphi_3 - \varphi_1) \frac{y}{L};$$

$$\Phi_1(y, \tau) = (\psi_2 - \psi_3) N_n - \left[\frac{\dot{\eta}_1}{k \gamma_0} + \frac{\delta}{\lambda} \eta_1 \right] \lambda \bar{F}_\delta + N_n (\psi_3 - \psi_1) \frac{y}{L},$$

and transform

$$\tilde{\bar{v}}_{nm} = \int_0^L \bar{v}_n \sin \varkappa_m \frac{y}{L} dy; \quad \varkappa_m = m\pi \ (m = 1, 2, \ldots, \infty) \tag{9-69}$$

converts the problem to the form

$$\frac{d\tilde{\bar{\vartheta}}_{2nm}}{d\tau} = -a \left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2} \right) \tilde{\bar{\vartheta}}_{2nm} + \frac{\varepsilon\rho}{c} \frac{d\tilde{\bar{\vartheta}}_{2nm}}{d\tau} + \tilde{\bar{L}}_{1nm}(\tau);$$

$$\frac{d\tilde{\bar{\vartheta}}_{2nm}}{d\tau} = -k_{\pi\pi} \left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2} \right) \tilde{\bar{\vartheta}}_{2nm} - k_{\pi\pi}\delta \left(\frac{\mu_n^2}{R^2} + \frac{\varkappa_m^2}{L^2} \right) \tilde{\bar{\vartheta}}_{2nm} - \tilde{\bar{L}}_{2nm}(\tau); \tag{9-70}$$

$$\tilde{\bar{\vartheta}}_{2nm}(0) = \tilde{\bar{f}}_{1nm} - N_n M_m \varphi_3(0) + N_n (\varphi_3(0) - \varphi_1(0)) \frac{\tilde{y}}{L} = \tilde{\bar{H}}_{1nm};$$

$$\tilde{\bar{\vartheta}}_{2nm}(0) = \tilde{\bar{f}}_{2nm} - N_n M_m \psi_3(0) + N_n (\psi_3(0) - \psi_1(0)) \frac{\tilde{y}}{L} = \tilde{\bar{H}}_{2nm}.$$

Here

$$\tilde{\bar{L}}_{1nm}(\tau) = \left[\frac{1}{c\gamma_0} q - \varphi'_3 + \frac{\varepsilon\rho}{c} (\psi_2 + \psi'_2 - \psi'_3) \right] N_n M_m +$$

$$+ \frac{a\mu_n^2}{R^2} \eta_1 \bar{F}_\delta M_m + N_n (\varphi'_3 - \varphi'_1 + \psi'_3 - \psi'_1) \frac{\tilde{y}}{L} +$$

$$+\frac{a\mu_n^2}{R^2}(\varphi_2-\varphi_3)N_nM_m-\left[\frac{\mathrm{Fe}}{L}(\eta_1+\eta'_1)+\frac{\varepsilon\rho}{k_m c\gamma_0}(\eta_{m1}+\right.$$

$$\left.+\eta'_{m1})\right]\lambda F_b M_m+\frac{a\mu_n^2}{R^2}N_n(\varphi_3-\varphi_1)\frac{\tilde{y}}{L};$$

$$\tilde{\overline{L}}_{2nm}(\tau)=\left(\frac{1}{\gamma_0}\omega+\psi'_1\right)N_nM_m-\frac{k_m\mu_n^2}{R^2}N_nM_m[(\psi_2-\psi_3)+$$

$$+\delta(\varphi_2-\varphi_3)]-\left[\frac{\delta}{L}(\eta_1-\eta'_1)+\frac{1}{k_m\gamma_0}(\eta_{m1}-\eta'_{m1})\right]\lambda F_b M_m+$$

$$+\frac{k_m\mu_n^2}{R^2}\frac{\eta_{m1}}{k_m\gamma_0}\lambda F_b M_m-N_n(\psi'_3-\psi'_1)\frac{\tilde{y}}{L}-$$

$$-\frac{k_m\mu_n^2}{R^2}N_n[(\psi_3-\psi_1)+\delta(\varphi_3-\varphi_1)]\frac{\tilde{y}}{L};$$

$$M_m=\int_0^L\sin\varkappa_m\frac{y}{L}\,dy=\frac{2L}{(2m-1)\pi};$$

$$\frac{\tilde{y}}{L}=(-1)^{m+1}\frac{L}{m\pi}.$$

The solution of the system of ordinary differential equations (9-69) under conditions (9-70), as before, will be performed by the d'Alambert method. Then, considering the corresponding inversion formulas for transformation of (9-26) and (9-69) (see § 3-3) and keeping in mind the substitutions performed, we find the final solution of the problem

$$u=\psi_3-(\psi_3-\psi_1)\frac{y}{L}+\frac{\delta L}{2(\nu_2^2-\nu_1^2)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)\times$$

$$\times\sin\frac{m\pi y}{L}\left\{P_{n2}\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{m^2\pi^2}{L^2}\right)k_m\nu_2^2\tau\right]-\right.$$

$$\left.-P_{n1}\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{m^2\pi^2}{L^2}\right)k_m\nu_1^2\tau\right]\right\}; \qquad (9\text{-}71)$$

$$T=\varphi_3-(\varphi_3-\varphi_1)\frac{y}{L}-\frac{L}{2(\nu_2^2-\nu_1^2)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\|U_0\|^2}U_0\left(\mu_n\frac{\xi}{R}\right)\times$$

$$\times\sin\frac{m\pi y}{L}\left\{\left(\nu_1^2-\mathrm{Fe}-\frac{1}{\mathrm{Lu}}\right)P_{n2}\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{m^2\pi^2}{L^2}\right)k_m\nu_2^2\tau\right]-\right.$$

$$\left.\left(\nu_2^2-\mathrm{Fe}-\frac{1}{\mathrm{Lu}}\right)P_{n1}\exp\left[-\left(\frac{\mu_n^2}{R^2}+\frac{m^2\pi^2}{L^2}\right)k_m\nu_1^2\tau\right]\right\}. \qquad (9\text{-}72)$$

where

$$P_{nj} = \tilde{\bar{H}}_{1nm} + \frac{1}{\delta}\left(\nu_j^2 - \mathrm{Fe} - \frac{1}{\mathrm{Lu}}\right)\tilde{\bar{H}}_{2nm} + \int_0^{\tau}\left\{\tilde{\bar{L}}_{1nm} - \frac{1}{\delta}\left(\nu_j^2 - \frac{1}{\mathrm{Lu}}\right)\tilde{\bar{L}}_{2nm}\right\}\exp\left[\left(\frac{\mu_n^2}{R^2} + \frac{m^2\pi^2}{L^2}\right)k_m\nu_j^2\tau\right]d\tau \; (j = 1, 2).$$

The symbols presented here are obvious from our previous exposition.

9-3. Solution of Generalized System of Equations of Related Heat and Mass Transfer in Concrete Bodies

Based on the finite rate of transfer of moisture observed in a capillary porous body in experiments, A. V. Lykov [72a] suggested a generalized rule of moisture conductivity which, for example, under isothermal conditions is written as follows:

$$W_m = -k_m\gamma_0 \operatorname{grad} u - \tau_{rm}\frac{\partial W_m}{\partial\tau}, \tag{9-73}$$

where W_m *is the moisture flux density vector;* τ_{rm} *is the period of moisture* exchange relaxation.

According to the estimates of A. V. Lykov [72a], the period of moisture exchange relaxation for a capillary porous body has the order of magnitude $(0.4\text{-}1.2)\cdot 10^{-4}$ s. Then the generalized system of bonded heat and mass transfer for concrete bodies becomes

$$\frac{\partial T}{\partial\tau} = a\nabla^2 T + \frac{\varepsilon\rho}{c}\frac{\partial u}{\partial\tau} + \frac{1}{c\gamma_0}q(\tau);$$

$$\tau_{rm}\frac{\partial^2 u}{\partial\tau^2} + \frac{\partial u}{\partial\tau} = k_m\nabla^2 u + k_m\delta\nabla^2 T - \frac{1}{\gamma_0}w(\tau). \tag{9-74}$$

We will solve this system with a one-dimensional temperature and moisture fields and boundary conditions of the first and second kind.

This gives us a system of differential equations

$$
\begin{aligned}
&\frac{\partial T}{\partial \tau} = a\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial T}{\partial \xi}\right)\right] + \frac{\varepsilon\rho}{c}\frac{\partial u}{\partial \tau} + \frac{1}{c\gamma_0}q(\tau);\\
&\tau_{rm}\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial \tau} = k_{m}\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial u}{\partial \xi}\right)\right] +\\
&\quad + k_{m}\delta\left[\frac{1}{\xi^{i}}\frac{\partial}{\partial \xi}\left(\xi^{i}\frac{\partial T}{\partial \xi}\right)\right] - \frac{1}{\gamma_0}w(\tau),
\end{aligned}
\tag{9-75}
$$

initial conditions

$$
T(\xi, 0) = f_1(\xi);\quad u(\xi, 0) = f_2(\xi);\quad \frac{\partial u(\xi, 0)}{\partial \tau} = f_3(\xi)
\tag{9-76}
$$

and boundary conditions such as (9-21).

As before, we perform the substitution (9-22) and apply the finite integral transform (9-26) to the result.

We produce

$$
\begin{aligned}
&\frac{d\bar{\theta}_n}{d\tau} - \frac{\varepsilon\rho}{c}\frac{d\bar{\vartheta}_n}{d\tau} + \frac{\Omega_n^2}{Lu}\bar{\theta}_n = -\bar{Q}_{1n}(\tau);\\
&\tau_{rm}\frac{d^2\bar{\vartheta}_n}{d\tau^2} + \frac{d\bar{\vartheta}_n}{d\tau} + \Omega_n^2\bar{\vartheta}_n + \delta\Omega_n^2\bar{T}_n = \bar{Q}_{2n}(\tau);\\
&\bar{\theta}_n(0) = \bar{\Phi}_{1n}(0) - \bar{f}_{1n} = \bar{H}_{1n},\quad \bar{\vartheta}_n(0) = \bar{\Phi}_{2n}(0) - \bar{f}_{2n} = \bar{H}_{2n};\\
&\frac{d\bar{\vartheta}_n(0)}{d\tau} = \bar{\Phi}'_{2n}(0) - \bar{f}_{3n} = \bar{H}_{3n},
\end{aligned}
$$

where

$$
\begin{aligned}
&\bar{Q}_{1n}(\tau) = \frac{1}{c\gamma_0}q(\tau)N_n - \bar{\Phi}'_{1n}(\tau) + \frac{\varepsilon\rho}{c}\bar{\Phi}'_{2n}(\tau);\\
&\bar{Q}_{2n}(\tau) = \frac{1}{\gamma_0}w(\tau)N_n + \bar{\Phi}'_{2n}(\tau) + \tau_{rm}\bar{\Phi}''_{2n}(\tau);\\
&\Omega_n^2 = \frac{k_{m}\mu_n^2}{R^2}.
\end{aligned}
$$

Let us now apply an integral Laplace transform

$$\tilde{\bar{v}}_n = \int_0^\infty \bar{v}_n e^{-s\tau} d\tau.$$

This yields

$$\left(s + \frac{\Omega_n^2}{\mathrm{Lu}}\right)\tilde{\bar{\theta}}_n - \frac{\varepsilon\rho}{c} s\tilde{\bar{\vartheta}}_n = -\tilde{\bar{Q}}_{1n} + \bar{\Pi}_{1n} - \frac{\varepsilon\rho}{c}\bar{\Pi}_{2n};$$

$$\Omega_n^2 \delta\tilde{\bar{\theta}}_n + (\tau_{rm}s^2 + s + \Omega_n^2)\tilde{\bar{\vartheta}}_n = \tilde{\bar{Q}}_{2n} + \tau_{rm}(s+1)\bar{\Pi}_{2n} + \tau_{rm}\bar{\Pi}_{3n}.$$

where

$$\tilde{\bar{\theta}}_n = \tau_{rm}(-\tilde{\bar{Q}}_{1n} + \bar{\Pi}_{1n})\frac{s^2}{\Delta(s)} + \left(-\tilde{\bar{Q}}_{1n} + \frac{\varepsilon\rho}{c}\tilde{\bar{Q}}_{2n} + \bar{\Pi}_{1n} - \right.$$
$$\left. - \frac{\varepsilon\rho}{c}(1-\tau_{rm})\bar{\Pi}_{2n} + \frac{\varepsilon\rho}{c}\tau_{rm}\bar{\Pi}_{3n}\right)\frac{s}{\Delta(s)} +$$
$$+ \Omega_n^2\left(-\tilde{\bar{Q}}_{1n} + \bar{\Pi}_{1n} - \frac{\varepsilon\rho}{c}\bar{\Pi}_{2n}\right)\frac{1}{\Delta(s)};$$

$$\tilde{\bar{\vartheta}}_n = \frac{\varepsilon\rho}{c}\tau_{rm}\bar{\Pi}_{2n}\frac{s^2}{\Delta(s)} + \left[\tilde{\bar{Q}}_{2n} + \tau_{rm}\left(\frac{\varepsilon\rho}{c}\frac{\Omega_n^2}{\mathrm{Lu}} + 1\right)\bar{\Pi}_{2n} + \right.$$
$$\left. + \tau_{rm}\bar{\Pi}_{3n}\right]\frac{s}{\Delta(s)} - \Omega_n^2\left[\delta\tilde{\bar{Q}}_{1n} - \frac{1}{\mathrm{Lu}}\tilde{\bar{Q}}_{2n} + \left(\mathrm{Fe} - \tau_{rm}\frac{1}{\mathrm{Lu}}\right)\bar{\Pi}_{2n} - \right.$$
$$\left. - \tau_{rm}\frac{1}{\mathrm{Lu}}\bar{\Pi}'_{3n} - \delta\bar{\Pi}_{1n}\right]\frac{1}{\Delta(s)},$$

where

$$\Delta(s) = \tau_{rm}s^3 + \left(1 + \tau_{rm}\frac{\Omega_n^2}{\mathrm{Lu}}\right)s^2 + \Omega_n^2\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)s + \frac{\Omega_n^2}{\mathrm{Lu}}.$$

Based on the second theorem of expansion, we find the originals $\bar{T}_n$ and $\bar{u}_n$:

$$\overline{\vartheta}_n=\sum_{\nu=1}^{3}\frac{e^{s_\nu \tau}}{\Delta'(s_\nu)}\Big\{(\tau_{rm}s_\nu^2+s_\nu+\Omega_n^2)\,\overline{\Pi}_{1n}-\frac{\varepsilon\rho}{c}[(1-\tau_{rm})s_\nu^2+$$

$$+\Omega_n^2]\,\overline{\Pi}_{2n}+\tau_{rm}s_\nu\frac{\varepsilon\rho}{c}\overline{\Pi}_{3n}-(\tau_{rm}s_\nu^2+s_\nu+\Omega_n^2)\times$$

$$\times\int_0^{\tau}\overline{Q}_{1n}(\tau)\,e^{-s_\nu\tau}\,d\tau-s_\nu\frac{\varepsilon\rho}{c}\int_0^{\tau}\overline{Q}_{2n}(\tau)\,e^{-s_\nu\tau}\,d\tau\Big\};$$

$$\overline{\vartheta}_n=\sum_{\nu=1}^{3}\frac{e^{s_\nu \tau}}{\Delta'(s_\nu)}\Big\{\Omega_n^2\delta\overline{\Pi}_{1n}+\Big[\tau_{rm}\frac{\varepsilon\rho}{c}s_\nu^2+\tau_{rm}\Big(1+$$

$$+\frac{\varepsilon\rho}{c}\frac{\Omega_n^2}{\mathrm{Lu}}\Big)s_\nu-\Omega_n^2\Big(\mathrm{Fe}-\tau_{rm}\frac{1}{\mathrm{Lu}}\Big)\Big]\overline{\Pi}_{2n}+$$

$$+\tau_{rm}\Big(s_\nu+\frac{\Omega_n^2}{\mathrm{Lu}}\Big)\overline{\Pi}_{3n}-\Omega_n^2\delta\int_0^{\tau}\overline{Q}_{1n}(\tau)\,e^{-s_\nu\tau}\,d\tau+$$

$$+\Big(s_\nu+\frac{\Omega_n^2}{\mathrm{Lu}}\Big)\int_0^{\tau}\overline{Q}_{2n}(\tau)\,e^{-s_\nu\tau}\,d\tau\Big\},$$

where s_ν is the root of the cubic equation

$$\Delta(s)=0;$$

$$\Delta'(s)=3\tau_{rm}s^2+2s\Big(1+\tau_{rm}\frac{\Omega_n^2}{\mathrm{Lu}}\Big)+\Omega_n^2\Big(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\Big). \qquad (9\text{-}77)$$

Let us define the roots of the cubic equation (9-77).

We can rewrite the equation as follows:

$$as^3+bs^2+cs+d=0.$$

Here

$$a=\tau_{rm};\ b=1+\tau_{rm}\frac{\Omega_n^2}{\mathrm{Lu}};\ c=\Omega_n^2\Big(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\Big);\ d=\frac{\Omega_n^4}{\mathrm{Lu}}.$$

We introduce

$$2g = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a};$$

$$3h = \frac{(3ac - b^2)}{3a^2}.$$

We can show that for concretes $g > 0$, $h > 0$ and the discriminant $D = g^2 + h^3 <$
< 0.

Then, as we know

$$s_1 = -2r\cos\frac{\varphi}{3} - \frac{b}{3a};$$

$$s_2 = +2r\cos\left(60^\circ - \frac{\varphi}{3}\right) - \frac{b}{3a};$$

$$s_3 = +2r\cos\left(60^\circ + \frac{\varphi}{3}\right) - \frac{b}{3a},$$

where $\cos\phi = g/r^3$, $r = \pm\sqrt{|h|}$, where the sign of r coincides with the sign of g.

Obviously

$$\cos\left(60^\circ + \frac{\varphi}{3}\right) \leqslant \frac{1}{2};\ r = \sqrt{\left|\frac{c}{3a} - \frac{b^2}{9a^2}\right|} < \frac{b}{3a}.$$

Therefore

$$s_3 < 0.$$

But

$$s_1 s_2 s_3 = -\frac{d}{a} < 0.$$

From this

$$s_2 < 0.$$

Formulas (9-78), considering the temporarily introduced symbols, also define the three real different and negative roots s_ν of third power equations (9-77).

Thus, the final solution to the problem

$$
\begin{aligned}
u = \Phi_2(\xi, \tau) - \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \sum_{\nu=1}^{3} \frac{e^{s_\nu \tau}}{\Delta'(s_\nu)} \Big\{ \Omega_n^2 \delta \overline{\Pi}_{1n} + \\
+ \left[\tau_{rm} \frac{\varepsilon\rho}{c} s_\nu^2 + \tau_{rm}\left(1 + \frac{\varepsilon\rho}{c} \frac{\Omega_n^2}{\mathrm{Lu}}\right) s_\nu - \Omega_n^2 \left(\mathrm{Fe} - \tau_{rm} \frac{1}{\mathrm{Lu}}\right)\right] \times \\
\times \overline{\Pi}_{2n} + \tau_{rm}\left(s_\nu + \frac{\Omega_n^2}{\mathrm{Lu}}\right) \overline{\Pi}_{3n} - \int_0^{\tau} \Big[\Omega_n^2 \delta \overline{Q}_{1n}(\tau) - \\
- \left(s_\nu + \frac{\Omega_n^2}{\mathrm{Lu}}\right) \overline{Q}_{2n}(\tau) \Big] e^{-s_\nu \tau} d\tau \Big\};
\end{aligned}
\tag{9-78}
$$

$$
\begin{aligned}
T = \Phi_1(\xi, \tau) - \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \sum_{\nu=1}^{3} \frac{e^{s_\nu \tau}}{\Delta'(s_\nu)} \Big\{ (\tau_{rm} s_\nu^2 + \\
+ s_\nu + \Omega_n^2) \overline{\Pi}_{1n} - \frac{\varepsilon\rho}{c} [(1 - \tau_{rm}) s_\nu + \Omega_n^2] \overline{\Pi}_{2n} + \\
+ \tau_{rm} s_\nu \frac{\varepsilon\rho}{c} \overline{\Pi}_{3n} - \int_0^{\tau} [(\tau_{rm} s_\nu^2 + s_\nu + \Omega_n^2) \overline{Q}_{1n}(\tau) + \\
+ s_\nu \frac{\varepsilon\rho}{c} \overline{Q}_{2n}(\tau) \Big] e^{-s_\nu \tau} d\tau \Big\}.
\end{aligned}
\tag{9-79}
$$

With boundary conditions of the second kind, the solution of the problem is

$$
\begin{aligned}
u = \frac{2^i}{(R_2^i + R_1^i)(R_2 - R_1)} \int_{R_1}^{R_2} \xi^i f_2(\xi) d\xi - \frac{1}{\gamma_0} \int_0^{\tau} w(\tau) d\tau - \\
- \frac{2^i}{\gamma_0 (R_2^i + R_1^i)(R_2 - R_1)} \int_0^{\tau} [R_1^i \eta_{m1}(\tau) + R_2^i \eta_{m2}(\tau)] d\tau + \\
+ \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \sum_{\nu=1}^{3} \frac{e^{s_\nu \tau}}{\Delta'(s_\nu)} \Big\{ \Omega_n^2 \delta \overline{f}_{1n} + \\
+ \left[\tau_{rm} \frac{\varepsilon\rho}{c} s_\nu^2 + \tau_{rm}\left(1 + \frac{\varepsilon\rho}{c} \frac{\Omega_n^2}{\mathrm{Lu}}\right) s_\nu - \Omega_n^2 \Big(\mathrm{Fe} - \right. \\
\left. - \tau_{rm} \frac{1}{\mathrm{Lu}}\Big)\right] \overline{f}_{2n} + \tau_{rm}\left(s_\nu + \frac{\Omega_n^2}{\mathrm{Lu}}\right) \overline{f}_{3n} + \frac{\Omega_n^2 \delta}{c\gamma_0} F(\tau, \eta, q) - \\
- \frac{1}{\gamma_0}\left(s_\nu + \frac{\Omega_n^2}{\mathrm{Lu}}\right) F(\tau, \eta_m, w) \Big\};
\end{aligned}
\tag{9-80}
$$

$$T = \frac{2^i}{(R_2^i + R_1^i)(R_2 - R_1)} \int_{R_1}^{R_2} \xi^i f_1(\xi)\, d\xi + \frac{1}{c\gamma_0} \int_0^{\tau} q(\tau)\, d\tau +$$

$$+ \frac{2^i}{c\gamma_0 (R_2^i + R_1^i)(R_2 - R_1)} \int_0^{\tau} \{[R_1^i \eta_1(\tau) + R_2^i \eta_2(\tau)] -$$

$$- \varepsilon\rho [R_1^i \eta_{m1}(\tau) + R_2^i \eta_{m2}(\tau)]\}\, d\tau +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\|U_0\|^2} U_0\left(\mu_n \frac{\xi}{R}\right) \sum_{\nu=1}^{3} \frac{e^{s_\nu \tau}}{\Delta'(s_\nu)} \Big\{ (\tau_{rm} s_\nu^2 + s_\nu + \Omega_n^2) \bar{f}_{1n} -$$

$$- \frac{\varepsilon\rho}{c} [(1 - \tau_{rm}) s_\nu + \Omega_n^2] \bar{f}_{2n} + \tau_{rm} \frac{\varepsilon\rho}{c} \bar{f}_{3n} + \frac{1}{c\gamma_0} (\tau_{rm} s_\nu^2 +$$

$$+ s_\nu + \Omega_n^2) F(\tau, \eta, q) - \frac{\varepsilon\rho}{c\gamma_0} s_\nu F(\tau, \eta_m, w) \Big\}, \qquad (9\text{-}81)$$

where

$$F(\tau, l, g) = \int_0^{\tau} \Big[R_1^i U_0\left(\mu_n \frac{R_1}{R}\right) l_1(\tau) +$$

$$+ R_2^i U_0\left(\mu_n \frac{R_2}{R}\right) l_2(\tau) + g(\tau) \Big] e^{-s_\nu \tau}\, d\tau \; (l = \eta, \eta_m;\ g = q, w).$$

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